THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

LESTER R. FORD, Editor-in-Chief

WITH THE CO-OPERATION OF

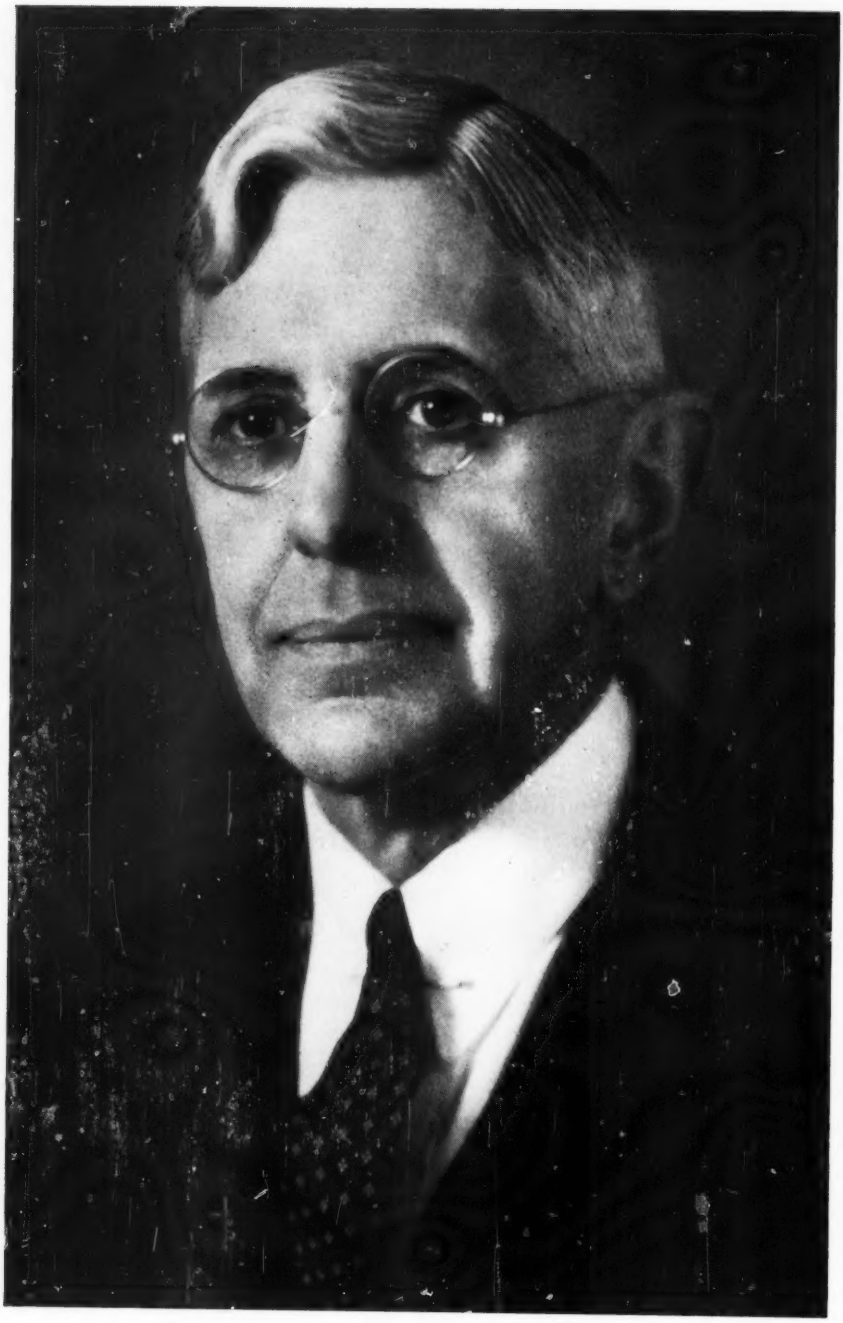
L. M. BLUMENTHAL
N. B. CONKWRIGHT
H. S. M. COXETER
W. M. DAVIS
OTTO DUNKEL
B. F. FINKEL
J. S. FRAME
ORRIN FRINK, JR.
MARJORIE GROVES
M. R. HESTENES
B. W. JONES
R. E. LANGER
J. R. MUSSELMAN
C. V. NEWSOM
VIRGIL SNYDER
MARIE J. WEISS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 50

1943

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND CHICAGO, ILL.



WILLIAM DEWEESE CAIRNS

OUR RETIRING SECRETARY-TREASURER

Twenty-seven years have now passed since the Mathematical Association of America was founded. During this time its membership has increased to considerably more than twice the eleven hundred charter members; meetings are held in twenty-two different sections throughout the country; and an ambitious program of periodical and book publication, and of sponsorship of other periodicals and of prize competitions, has come to fruition. Throughout this period of increasing usefulness the now retiring Secretary-Treasurer, WILLIAM DEWEESE CAIRNS has been an outstanding leader in the determination of policy and in the conduct of all of the Association's affairs. In the extent and value of his services to the Association, he is rivalled only by Herbert Ellsworth Slaught (1861–1937), also an enthusiastic and constant worker for the interests of the Association from the time of its founding. To these two men far more than to any others the Association owes its success and its present strong position.

It must be the sincere hope of all who have the best interests of the Association at heart that Professor Cairns may long continue to contribute, by his influence and from the fullness of his wisdom, to her interests. For many decades to come his high achievement, his constructive leadership, and his great devotion will be remembered, and will serve as an inspiration to younger men who desire to advance the cause of mathematics in America.

EARL RAYMOND HEDRICK

November 1942

Since these words were written Professor Cairns has been elected to the presidency of the Association. It is a fortunate circumstance for us that he will guide the destinies of our organization during the difficult days that lie ahead.

L.R.F.

WHAT IS DIMENSION?

KARL MENGER, University of Notre Dame

1. Solids, surfaces, and lines. Strictly speaking, all material objects are 3-dimensional. Yet, only such objects as a metal sphere, a wooden block, or a rock are considered to be typical representatives of 3-dimensional entities (solids). A piece of sheet-iron, paper, and a membrane approach what we mean when we speak of 2-dimensional objects (surfaces). Wire, threads, and streaks of chalk represent our idea of 1-dimensional entities (lines).

What is the difference between objects of different dimensions? Originally, mathematicians believed it to be a difference in quantity, in the sense that a surface contains more points than a line and less points than a solid. Now primarily the words "more," "less," and "equally many" are restricted to finite sets while surfaces, as well as lines and solids, contain infinitely many points. But Georg Cantor extended their use to all sets. We say that two sets—finite or infinite—contain equally many elements if we can establish a one-to-one correspondence between their elements. Cantor found that two infinite totalities do not necessarily contain equally many elements. For instance, among geometrical objects a straight line segment contains more points than some dispersed infinite sets, *e.g.*, the set of all points on a straight line whose distances from a certain point are integers. However, a straight line segment, a square, and a cube do contain equally many points [1]. Since these objects are of different dimensions, it follows that dimension is not a quantitative property.

Later, geometers thought that the difference between a 1-dimensional and a higher-dimensional object lay in the fact that the former, but not the latter, can be traversed by a continuously moving point. Indeed, lines on a paper or a blackboard are drawn, *i.e.*, traversed by the point of a pencil or chalk. However, Peano found that a continuously moving point can traverse a square surface or a solid cube though nobody would call these objects 1-dimensional. On the other hand, 1-dimensional objects were found which cannot be traversed by a continuously moving point [2]. The fact that an object is the path of a point is interesting in itself, but has no bearing on the question of the dimension of the object [3].

When one-to-one, as well as continuous, mappings had proved to be inadequate bases for the definition of dimension, mathematicians attempted to characterize dimension of a totality T as the least number of real numbers required to describe topologically (in a one-to-one and bi-continuous way) the elements of T. Each point of our ordinary space can be topologically characterized by three, but not less than three, real numbers, *e.g.*, its Cartesian or spherical coordinates; each point of a simple surface by two, but not less than two, real numbers, *e.g.*, the points of a sphere by longitude and latitude; each point of a simple line by one number. Thus, by the last definition, our space is 3-dimensional, simple surfaces are 2-dimensional, simple lines are 1-dimensional. Similarly, each color sensation of a normal eye can be topologically characterized by

three, but not less than three, real numbers, *viz.*, the quantities of three standard colors whose mixture produces an identical sensation. Hence, the totality of color sensations of a normal eye is 3-dimensional while the corresponding totalities for a partially or totally color blind eye are but 2- and 1-dimensional, respectively. In the same way, a totality of all mixtures of four ingredients which cannot be obtained by mixing less than four of them is called four-dimensional. In fact, in this direction lies our only elementary analytical approach to the fourth dimension and higher-dimensional spaces.

Unfortunately, however, the last definition applies only to very simple spatial entities, *viz.*, to those which can be obtained by means of a very simple transformation from a straight segment, a square, or a cube. Such entities are called arcs, discs, and topological spheres. In our space and in the plane, arcs and discs form only a small part of the lines and surfaces studied by modern geometry. Even if we admit objects which are sums of a finite number of arcs and discs our domain is still very restricted. For instance, the line mentioned above, which cannot be traversed by a continuously moving point [2], does not belong to this domain since it is not a sum of a finite number of arcs. In fact, it is the sum of infinitely many arcs, but all sets which are sums of infinitely many arcs cannot possibly be called 1-dimensional since the square and the cube are sums of infinitely many straight segments. [4].

To formulate the intuitive difference between lines, surfaces, and solids one can devise a simple experiment whose outcome depends upon the dimension of the object to which it is applied [5]. We cut out from the object a piece surrounding a given point. If the object is a solid we need a saw to accomplish this, and the cutting is along surfaces. If the object is a surface a pair of scissors suffices, and the cuts are along curves. If we deal with a curve we may use a pair of pliers and have to pinch the object in dispersed points. Finally, in a dispersed object no tool is required to perform our experiment, since nothing needs to be dissected. This characterization of dimension leads from n-dimensional to $(n-1)$-dimensional objects. It ends with dispersed sets, naturally called 0-dimensional, and, beyond these, with "nothing," in set theory called the "vacuous set." It is, therefore, convenient to consider the latter as -1-dimensional.

2. The definition of dimension. To make this idea precise we need only two simple auxiliary concepts: neighborhood and boundary. In our space we call a set N a *neighborhood* if each point of N is center of a sphere (though perhaps a very small sphere) all of whose points belong to N. The interior of a cube is a neighborhood, whereas a cube with its faces is not. For even the smallest sphere about a point of a face contains points not belonging to the cube. Nor is a plane a neighborhood in our space. For each sphere about each point of a plane contains points not belonging to the plane. The *boundary* of a neighborhood N is the set of all points which do not belong to N but are centers of arbitrarily small spheres which contain some points of N. For the interior of the cube the boundary obviously consists just of the six faces.

In terms of these concepts the result of our recursive dimension experiment can be explained as follows: A set S of points of our space is *at most n-dimensional* if each point of S lies in arbitrarily small neighborhoods whose boundaries have at most $(n-1)$-dimensional intersections with S. The set S is *n-dimensional* if it is at most n-dimensional but not at most $(n-1)$-dimensional. That S is not at most $(n-1)$-dimensional means that S contains at least one point at which S is at least n-dimensional, that is to say, a point which does not lie in arbitrarily small neighborhoods whose boundaries have at most $(n-2)$-dimensional intersections with S; the boundaries of all sufficiently small neighborhoods of such a point have at least $(n-1)$-dimensional intersections with S. The vacuous set, called -1-dimensional, is the starting point of the recursive definition [6].

By this definition, a set S is 0-dimensional if it is not vacuous, and each point of S lies in arbitrarily small neighborhoods whose boundaries have -1-dimensional, *i.e.*, vacuous, intersections with S—in other words, no points in common with S. A set S is 1-dimensional if it is not 0-dimensional and each point lies in arbitrarily small neighborhoods whose boundaries have at most 0-dimensional intersections with S. But it should be clearly understood that a point of a 1-dimensional set S may also be contained in arbitrarily small neighborhoods whose boundaries have more than 0-dimensional intersections with S. For instance, each point of a straight line S is contained in arbitrarily small neighborhoods whose boundaries contain whole pieces of S. Such neighborhoods can be formed by adding two cubes of different size, one of which has a face passing through S.

Furthermore, it should be clear that we cannot always expect to find *simple* neighborhoods of a point of an n-dimensional set whose boundaries have at most $(n-1)$-dimensional intersections with S. One of the most interesting examples in this respect arises from the study of the following four sets whose sum incidentally exhausts our space:

the set S_0 of all points which have three irrational coordinates,
the set S_1 of all points which have one rational and two irrational coordinates,
the set S_2, of all points which have two rational and one irrational coordinate,
the set S_3 of all points which have three rational coordinates.

If a, b, c are any three rational constants, then the planes $x=a$, $y=b$, $z=c$ do not contain any points of S_0, and the planes $ax+by+cz=1$ do not contain any points of S_2. If α, β, γ are any three irrational constants, then the planes $x=\alpha$, $y=\beta$, $z=\gamma$ do not contain any points of S_3. Now, for $i=0, 2, 3$, each point of S_i is contained in arbitrarily small *cubes* whose faces are part of such planes which have no point in common with S_i. Hence, S_0, S_2, S_3 are 0-dimensional. So is S_1 but the proof of this fact is much more difficult [7]. For not only each plane meets S_1, but as Schreier noticed, each surface of the form $z=f(x, y)$ where f is a continuous function, has points in common with S_1 and the same is true for each surface $y=f(x, z)$ and $x=f(y, z)$. In fact, only recently S. G. Reed, Jr. and the author constructed [8] a neighborhood whose necessarily complicated boundary has no point in common with S_1.

Since dimension of the subsets of our space has been defined in terms of neighborhoods the definition is applicable to the subsets of all spaces in which neighborhoods are given. An example of such a space is the 4-dimensional euclidean space whose points are the quadruples of real numbers x, y, z, u and in which the sphere with radius r and center x_0, y_0, z_0, u_0 consists of the points x, y, z, u satisfying the inequality $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(u-u_0)^2=r^2$. A set N is a neighborhood if each point of N is center of a sphere all of whose points belong to N.

3. Criteria for a satisfactory definition. Now let us examine the definition of dimension. Its objective is to make precise and to extend the ordinary usage of the words "1-dimensional," "2-dimensional," and "3-dimensional." *A good definition of a word must include all entities which are always denoted and must exclude all entities which are never denoted by the word.* For the word "1-dimensional" straight lines, ellipses, and lemniscates are objects of the former type; square surfaces, solid cubes, and finite sets of the latter type. *A good definition should extend the use of the word by dealing with objects not known or not dealt with in ordinary language. With regard to such entities, a definition cannot help being arbitrary.* In connection with the word "1-dimensional" consider the four sets S_i whose sum exhausts our space. A general definition of "1-dimensional" will imply for each of the sets S_0, S_0+S_1, $S_0+S_1+S_2$ whether or not it is 1-dimensional. Our definition implicitly assigns to these sets the dimensions 0, 1, and 2, respectively, which like each assignment is somewhat arbitrary since ordinary language does not assign to them any dimension. *A good definition must yield many consequences, in particular theorems which are aesthetically satisfactory by their generality and simplicity, and theorems connecting the defined concept with concepts of other theories.* It is these theorems which justify the unavoidable arbitrary element of the definition. Some of the theorems will extend statements which are true in the restricted domain of ordinary language to the extended domain of the definition. Other theorems will exhibit interesting exceptions or even correct erroneous habits of thinking.

The definition outlined in this paper has yielded an extensive dimension theory which, since its foundation in the early twenties, has developed into one of the central branches of topology. Since even an enumeration of the main results would surpass the limits of this paper we shall confine ourselves to a few illustrations of the general criteria of the previous paragraph. An example of the numerous statements extending to all sets a proposition known to hold for the simple objects of ordinary language, is the theorem [5] that an n-dimensional set S contains infinitely many points at which S is n-dimensional, and that these points form a set S' which is at least $(n-1)$-dimensional. Under certain conditions we can say that S' is n-dimensional. However, there are rather unexpected exceptions in which S' is only $(n-1)$-dimensional. One of the facts which justify our definition of 0-dimensionality is the simple and beautiful general theorem that each n-dimensional set is the sum of $n+1$ but not less than $n+1$ 0-dimen-

sional sets. If we had assigned to the sets S_0, S_0+S_1, $S_0+S_1+S_2$ other dimensions than we did, it would have been at the expense of a simple systematic theory.

4. Five properties of dimension. In concluding, I shall select five of the theorems of dimension theory which, as we shall see, are of a particular importance:

I. *The euclidean n-space is n-dimensional.* (This theorem is due to Brouwer.) The cases $n=1, 2, 3$ of this theorem show, in particular, that the definition of 1-dimensionality excludes square surfaces and solid cubes which ordinary language always excludes and which older definitions failed to exclude.

II. *The topological image of an n-dimensional set is n-dimensional.* In conjunction with theorem I this simple theorem shows that the concepts of 1-dimensional and 2-dimensional sets include arcs and discs which are always called 1-dimensional and 2-dimensional, respectively.

III. *Each part of an n-dimensional set is at most n-dimensional.* Natural and simple as this theorem is it does not hold for some other definitions of dimension [9].

IV. *A set S cannot be split into denumerably many* [4] *closed* [10] *summands each of which is of smaller dimension than S.* (This so-called sum-theorem which occupies a central role in dimension theory, as well as the simple theorems II and III are due to Urysohn and the author.)

V. *Each n-dimensional set can be topologically transformed into a subset of a compact* [10] *n-dimensional set.* (This theorem is due to Hurewicz.)

5. Further aspects of the problem. What is dimension? Have we answered this question? In one sense, we have. We have explained which sets are 1-dimensional, which are 2-dimensional, etc. In fact, with each subset of our space and with each subset of much more general spaces we have associated an integer, the dimension of the set. This is also expressed by saying that dimension is a set function. However, there are many other set functions. With each set in our space we may, for example, associate the number of pieces of which it consists, or its measure (in some sense). In this connection the question "What is dimension?" may be interpreted in the following sense: "Among the many set functions, by which properties is dimension characterized?"

So far this question has only been answered for the plane [11]. There dimension is characterized by the properties described in theorems I to V, that is to say: In the plane, dimension is the only set function with the following properties:

1) It assumes the values 2, 1, 0, -1 for the square, the straight line segment, the single point, and the vacuous set, respectively;

2) It assumes the same value for any two sets which can be obtained from each other by a topological transformation;

3) It never has a greater value for the part than for the whole;

4) No set can be split into denumerably many closed sets of smaller function value;

5) Each set can be topologically transformed into a part of a compact set of equal function value.

In the plane, therefore, this is another answer to the question, "What is dimension?"

Footnotes

1. Also some dispersed sets and a straight line segment contain equally many points, *e.g.*, the set of all points on a line whose distance from a certain point is irrational, or Cantor's so-called *discontinuum*.

2. *E.g.*, the so-called *sinusoid* consisting of the points (x, y) of the plane for which either $0<x\leqq 1$ and $y=\sin 1/x$ or $x=0$ and $-1\leqq y\leqq 1$.

3. In this connection we may mention a comparatively recent result of the theory of curves. If a set S as well as each subcontinuum of S can be traversed by a continuously moving point, then S is 1-dimensional in the sense defined in this paper. The converse of this theorem is not true.

4. One might think that 1-dimensional are the sets which are the sum of *denumerably many* arcs, *i.e.*, of as many arcs as there are integers. But this definition would still be too narrow while the class of entities which are sums of *non*-denumerably many arcs contains the square and the cube and thus is too wide.

5. See the author's book "Dimensionstheorie" 1928.

6. The history of this definition and the ensuing theory is outlined in the beautiful exposition of Hurewicz and Wallman, Dimension Theory, Princeton University Press, 1941.

7. See Hurewicz and Wallman, p. 19.

8. To be published in Issue 5 of the Reports of a Mathematical Colloquium, University of Notre Dame publication.

9. See the Appendix to Hurewicz and Wallman, Dimension Theory.

10. A set C is *closed* if its complement is a neighborhood, and hence C contains all cluster points of C, *i.e.*, all points of which each neighborhood has infinitely many points in common with C. A set C is called *compact* if for each infinite subset of C there exists a cluster point in C. It should be noted that theorem IV, would not hold if we omitted the word closed: Our 3-dimensional space can be split into a finite number of sets of smaller dimensions which are not closed, *e.g.*, into the four 0-dimensional sets S_0, S_1, S_2, S_3. Nor would theorem IV hold if we admitted splitting in more than denumerably many closed sets. Our 3-dimensional space is sum of infinitely many (but not denumerably many) closed 0-dimensional sets *e.g.*, of sets each of which consists of exactly one point.

11. Monatshefte f. Mathematik u. Physik, 36, 1929, p. 193.

DIVISORS OF ZERO IN POLYNOMIAL RINGS

ALEXANDRA FORSYTHE, Vassar College

This is an alternative proof to Theorem 2 of an article by N. H. McCoy* on divisors of zero. The definitions are those of Professor McCoy unless otherwise stated.

We postulate a commutative ring R and let $R[x]$ be the commutative extension ring obtained through the adjunction of an indeterminate x to the ring R. We shall say that an element $f(x)$ is a *divisor of zero* in $R[x]$ if there exists a non-zero element $g(x)$ of $R[x]$ such that the product $g(x)f(x)=0$.

* McCoy, N. H., Remarks on divisors of zero, this MONTHLY, vol. 49, May 1942, pp. 286–295.

THEOREM. (McCoy) *If $f(x)$ is a divisor of zero in $R[x]$, then there exists a non-zero element c of R such that $c \cdot f(x) = 0$.*

By definition of divisor of zero, there exists in $R[x]$ a non-zero element $g(x)$ of some degree N such that $g(x)f(x)$ equals zero. Now by the zero degree elements, that is those of the form ax^0, in $R[x]$ we mean exactly the elements of the ring R. Hence McCoy's theorem will be proved if we prove:

THEOREM A. *For every such element $g(x)$ (defined just above) of degree N, $N>0$, there exists another element $g'(x)$ in $R[x]$ satisfying exactly the same conditions as $g(x)$ and of degree less than or equal to $N-1$.*

Proof. We may suppose $f(x)=\sum_{i=0}^{i=M} a_i x^i$ and $g(x)=\sum_{j=0}^{j=N} b_j x^j$, where the a_i and b_j are elements of R such that a_M and b_N are not zero. By hypothesis $g(x)f(x)=0$. Now if $b_0=0$ we have

$$g(x)f(x) = \sum_{j=0}^{j=N} b_j x^j f(x) = x \sum_{j=0}^{j=N-1} b_{j+1} x^j f(x) = 0.$$

In this case of course $g'(x)$ may be taken to be $\sum_{j=0}^{j=N-1} b_{j+1}x^j$. This expression may easily be seen to be different from zero since its coefficients include b_N. Its degree is $N-1$ and $g'(x)f(x)=0$.

Otherwise $b_0 \neq 0$ and, by equating coefficients in the expression $g(x)f(x)=0$, one has $b_0 a_0 = 0$. Since the original ring R is commutative $a_0 b_0$ is also zero. Consider the product $a_0 g(x)$.

$$a_0 g(x)f(x) = a_0 \sum_{j=0}^{j=N} b_j x^j f(x) = a_0 x \sum_{j=0}^{j=N-1} b_{j+1} x^j f(x) = 0.$$

If $a_0 g(x)$ is not equal to zero, the product $a_0 \sum_{j=0}^{j=N-1} b_{j+1}x^j$ can easily be seen to be different from zero and may be taken as $g'(x)$.

If $a_0 g(x)=0$, look at $a_i g(x)$ $(0 \leqq i < M)$. If all these products are zero, b_0 satisfies the conditions of $g'(x)$ for, since $a_i g(x)=0$ $(0 \leqq i < M)$, it follows that $a_i b_0 = 0$ $(0 \leqq i < M)$. Then this together with the fact that $g(x)f(x)=0$, gives us that $b_0 a_M = a_M b_0 = 0$. Hence $b_0 f(x)=0$.

If $a_0 g(x)=0$ and $a_i g(x)=0$ $(0 \leqq i < p' < M)$ but $a_{p'} g(x) \neq 0$, it follows in the same way as before that $a_i b_0 = 0 (0 \leqq i \leqq p')$. Then one has

$$a_{p'} g(x)f(x) = a_{p'} \sum_{j=0}^{j=N} b_j x^j f(x) = a_{p'} x \sum_{j=0}^{j=N-1} b_{j+1} x^j f(x) = 0.$$

We may take $a_{p'} \sum_{j=0}^{j=N-1} b_{j+1}x^j$ as $g'(x)$. Again this element is non-zero, it has degree $N-1$ and it satisfies the condition $g'(x)f(x)=0$. This completes the proof of Theorem A and hence of McCoy's theorem.

THE DERIVATIVES OF COMPOSITE FUNCTIONS

ARNOLD DRESDEN, Swarthmore College

1. Introduction. In Vol. I of the *Quarterly Journal of Mathematics*, Faa de Bruno gave an explicit formula for the nth derivative of a composite function.* It is the purpose of the present note to obtain this formula, in a slightly modified form which has proved to be convenient in certain applications and the proof of which presents some interest.

2. Notation and result. Let $u=f(y)$, $y=g(x)$ and $u=f[g(x)]=F(x)$, represent single-valued and indefinitely differentiable functions of their arguments when $a \leqq x \leqq b$. Then we shall prove that

$$F^{(n)}(x) = \sum_{0}^{n-1} A_k^n f^{(n-k)}[g(x)], \tag{1}$$

where

$$A_k^n = \sum_{R_{n,k}} c_{p_1 \cdots p_{k+1}}^{nk} (g')^{p_1} (g'')^{p_2} \cdots (g^{(k+1)})^{p_{k+1}}; \tag{2}$$

the range $R_{n,k}$, $k=0, 1, \cdots, n-1$, of the summation extends over all sets of non-negative integers $p_1, \cdots, p_{k+1}$ which satisfy the conditions

$$R_{n,k}: \begin{cases} p_1 + p_2 + \cdots + p_{k+1} = n - k, \\ p_1 + 2p_2 + \cdots + (k+1)p_{k+1} = n. \end{cases} \tag{3}$$

Finally, for any solution $p_1, \cdots, p_{n+1}$ of (3), for given n and k, the coefficients $c_{p_1 \cdots p_{k+1}}^{nk}$ are integers represented by the formula

$$c_{p_1 \cdots p_{k+1}}^{nk} = \frac{n!}{p_1! \cdots p_{k+1}! (2!)^{p_2} \cdots [(k+1)!]^{p_{k+1}}}. \tag{4}$$

For example, the determination of $F^{(5)}$ requires the calculation of A_0^5, $A_1^5, \cdots, A_4^5$. To illustrate the procedure we shall calculate A_k^n, $k=0, \cdots, 4$. For A_0^n, the equations (3) reduced to $p_1=n$, so that

$$c_n^{n0} = \frac{n!}{n!} = 1 \quad \text{and} \quad A_0^n = (g')^n. \tag{5}$$

For A_1^n, we have the conditions $p_1+p_2=n-1$, $p_1+2p_2=n$, so that the range $R_{n,1}$ consists of the single set $p_1=n-2$, $p_2=1$, and

$$c_{n-2,1}^{n1} = \frac{n!}{(n-2)!2!} = \binom{n}{2} \quad \text{and} \quad A_1^n = \binom{n}{2}(g')^{n-2} g''. \tag{5a}$$

The coefficient A_2^n depends on the conditions $p_1+p_2+p_3=n-2$, $p_1+2p_2+3p_3=n$, whence we find $p_2+2p_3=2$. The range $R_{n,2}$ consists of the sets $p_1=n-3$,

* Compare also Goursat, Cours d'Analyse, I, 1910, p. 81.

$p_2=0$, $p_3=1$ and $p_1=n-4$, $p_2=2$, $p_3=0$, so that

$$A_2^n = \frac{n!}{(n-3)!3!}(g')^{n-3}g''' + \frac{n!}{(n-2)!2!(2!)^2}(g')^{n-4}(g'')^2$$

$$= \binom{n}{3}(g')^{n-3}g''' + 3\binom{n}{4}(g')^{n-4}(g'')^2.$$

In similar manner, we find

$$A_3^n = \binom{n}{4}(g')^{n-4}g^{IV} + 10\binom{n}{5}(g')^{n-5}g''g''' + 15\binom{n}{6}(g')^{n-6}(g'')^3,$$

$$A_4^n = \binom{n}{5}(g')^{n-5}g^{V} + 10\binom{n}{6}(g')^{n-6}(g''')^2 + 15\binom{n}{6}(g')^{n-6}g''g^{IV}$$

$$+ 105\binom{n}{7}(g')^{n-7}(g'')^2g''' + 105\binom{n}{8}(g')^{n-8}(g'')^4.$$

Thus we obtain

$$F^{(5)} = f^{V}(g')^5 + 10f^{IV}(g')^3g'' + f'''[10(g')^2g''' + 15g'(g'')^2]$$
$$+ f''[5g'g^{IV} + 10g''g'''] + f'g^{V}.$$

This result is readily verified. We observe that $R_{n,n-1}$ consists of the single set $0, \cdots, 0, 1$, so that

$$c_{0,\cdots,0,1}^{n,n-1} = 1 \quad \text{and} \quad A_{n-1}^n = g^{(n)}. \tag{6}$$

3. Proof of formula. The proof of our formula (1) will be made by induction. Its validity for $n=1$ follows from (5); the example discussed above shows moreover that it holds for $n=5$.

By differentiating (1) with respect to x, we obtain

$$F^{(n+1)}(x) = \sum_0^{n-1} A_k^n f^{(n-k+1)}(y)\cdot g' + \sum_0^{n-1}(A_k^n)'f^{(n-k)}(y)$$

$$= A_0^n f^{(n+1)}g' + \sum_1^{n-1}(A_kg' + A_{k-1}^{n\prime})f^{(n-k+1)} + A_{n-1}^{n\prime}f'.$$

Therefore, we have to show that

$$A_0^n g' = A_0^{n+1}, \quad A_{n-1}^{n\prime} = A_n^{n+1}, \quad \text{and} \tag{7}$$

$$A_k^n g' + A_{k-1}^{n\prime} = A_k^{n+1} \qquad \text{for } k = 1, \cdots, n-1. \tag{8}$$

The proof of (7) is contained in (5) and (6). In virtue of (2), the assertion in (8) is equivalent to

$$\sum_{R_{n,k}} c^{n,k}_{p_1\cdots p_{k+1}}(g')^{p_1+1}(g'')^{p_2}\cdots(g^{(k+1)})^{p_{k+1}}$$

$$+\sum_{R_{n,k-1}} c^{n,k-1}_{p_1\cdots p_k}[(g')^{p_1}\cdots(g^{(k)})^{p_k}]' \tag{9}$$

$$=\sum_{R_{n+1,k}} c^{n+1,k}_{p_1\cdots p_{k+1}}(g')^{p_1}\cdots(g^{(k+1)})^{p_{k+1}}, \qquad \text{for } k=1,\cdots,n-1.$$

According to (3), the range $R_{n+1,k}$ occurring on the right of (9) is determined by the equations

$$\begin{cases} p_1+\cdots+p_{k+1}=n-k+1 \\ p_1+2p_2+\cdots+(k+1)p_{k+1}=n+1. \end{cases} \tag{10}$$

In a set of non-negative integers which satisfy these conditions, we must have $p_{k+1}\leqq 1$, since it follows from equations (10) that $p_2+2p_3+\cdots+kp_{k+1}=k$; moreover if $p_{k+1}=1$, then $p_2=\cdots=p_k=0$, so that the only solution of (10) in which $p_{k+1}\neq 0$, is given by $p_1=n-k$, $p_2=\cdots=p_k=0$, $p_{k+1}=1$. Hence the only term on the right of (9), which actually involves $g^{(k+1)}$ is $c^{n+1,k}_{n-k,0,\cdots,0,1}(g')^{n-k}g^{(k+1)}$. The same considerations show that in $R_{n,k}$ the only term which involves $g^{(k+1)}$ is obtained for $p_1=n-k-1$, $p_2=\cdots=p_k=0$, $p_{k+1}=1$; hence in the first sum on the left, the only term which involves $g^{(k+1)}$ is $c^{n,k}_{n-k-1,0,\cdots,0,1}(g')^{n-k}g^{(k+1)}$. To obtain a term involving $g^{(k+1)}$ in the second sum on the left, we must have $p_k=1$ and hence $p_1=n-k$, $p_2=\cdots=p_{k-1}=0$; it follows that the only term of this type in the second sum is $c^{n,k-1}_{n-k,0,\cdots,0,1}(g')^{n-k}g^{(k+1)}$. Consequently, it remains to show that

$$c^{n,k}_{n-k-1,0,\cdots,0,1}+c^{n,k-1}_{n-k,0,\cdots,0,1}=c^{n+1,k}_{n-k,0,\cdots,0,1} \qquad \text{for } k=1,\cdots,n-1. \tag{11}$$

In virtue of (4), this reduces to

$$\frac{n!}{(n-k-1)!(k+1)!}+\frac{n!}{(n-k)!k!}=\frac{(n+1)!}{(n-k)!(k+1)!},$$

which is the well-known relation $\binom{n}{k+1}+\binom{n}{k}=\binom{n+1}{k+1}$ between binomial coefficients. Thus we have shown that the terms on the two sides of (9) which actually contain $g^{(k+1)}$ have equal coefficients.

It remains to consider those terms in (9) in which $g^{(k+1)}$ does not occur, *i.e.* the sets in $R_{n,k}$ and in $R_{n+1,k}$ in which $p_{k+1}=0$, and the sets in $R_{n,k-1}$ in which $p_k=0$. Let $p_1,\cdots,p_k,0$ be a set in the range $R_{n+1,k}$ so that the term $c^{n+1,k}_{p_1,\cdots,p_k,0}(g')^{p_1}\cdots(g^{(k)})^{p_k}$ occurs on the right of (9). Then $p_1-1,p_2,\cdots,p_k,0$ will be a set in the range $R_{n,k}$ and hence $c^{n,k}_{p_1-1,p_2,\cdots,p_k,0}(g')^{p_1}\cdots(g^{(k)})^{p_k}$ will occur in the first sum on the left of (9). Furthermore, $p_1,\cdots,p_{i-1},p_i+1,p_{i+1}-1,p_{i+2},\cdots,p_k$ will belong to the range $R_{n,k-1}$ for $i=1,\cdots,k-1$ and the terms $(p_i+1)c^{n,k-1}_{p_1,\cdots,p_{i-1},p_i+1,p_{i+1}-1,p_{i+2},\cdots,p_k}(g')^{p_1}\cdots(g^{(k)})^{p_k}$, $i=1,\cdots,k-1$ will occur in the second sum on the left, it being understood that any $c^{n,k}_{p_1,\cdots,p_{k+1}}$ in which one or more subscripts are negative, is to be replaced by zero. Con-

versely, to any set $p_1, \cdots, p_k, 0$ of $R_{n,k}$ correspond the set $p_1+1, p_2, \cdots, p_k, 0$ of $R_{n+1,k}$ and the sets $p_1, \cdots, p_{i-1}, p_i+1, p_{i+1}-1, p_{i+2}, \cdots, p_k$ of $R_{n,k-1}$. Therefore, it remains to show that, if $p_1, \cdots, p_k, 0$ is any set which belongs to $R_{n+1,k}$, then

$$(12) \qquad c^{n,k}_{p_1-1,p_2,\cdots,p_k,0} + \sum_{i=1}^{k-1} (p_i+1) c^{n,k-1}_{p_1,\cdots,p_{i-1},p_i+1,p_{i+1}-1,p_{i+2},\cdots,p_k} = c^{n+1,k}_{p_1\cdots p_k,0}.$$

By means of (4), it is found that the left side of this relation is equal to

$$\frac{n!}{(p_1-1)!p_2!\cdots p_k!(2!)^{p_2}\cdots(k!)^{p_k}}$$

$$+\sum_{i=1}^{k-1} \frac{(p_i+1)n!}{p_1!\cdots p_{i-1}!(p_i+1)!(p_{i+1}-1)!p_{i+2}!\cdots p_k!(2!)^{p_2}\cdots[(i-1)!]^{p_{i-2}}(i!)^{p_i+1}[(i+1)!]^{p_{i+1}-1}[(i+2)!]^{p_{i+2}}\cdots(k!)^{p_k}}$$

$$= \frac{n!}{p_1!\cdots p_k!(2!)^{p_2}\cdots(k!)^{p_k}}\left[p_1+\sum_1^{k-1}\frac{p_{i+1}(i+1)!}{i!}\right]$$

$$= \frac{(n+1)!}{p_1!\cdots p_k!(2!)^{p_2}\cdots(k!)^{p_k}} = c^{n+1,k}_{p_1,\cdots,p_k,0}, \quad \text{by use of (10) and (4).}$$

This completes the proof of (9) and hence the proof of (1). That the coefficients $c^{n,k}_{p_1,\cdots,p_{k+1}}$ are integers follows from (11) and (12), in conjunction with (5), (5a) and (6), by means of induction.

We observe that the coefficient A^n_k, defined by (2) is equal to $n!/(n-k)!$ times the terms of weight n in the expansion of $(g'+g''/2!+\cdots+g^{(k+1)}/(k+1))^{(n-k)}$ the derivatives $g^{(i)}$, $i=1, \cdots, k+1$ being given weight i.

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The eighteenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the University of Pittsburgh, Pittsburgh, Pennsylvania, on April 25, 1942. The chairman of the Section, Dr. R. G. Sturm of the Aluminum Research Laboratories, presided at both morning and afternoon sessions.

The attendance was fifty-three, including the following twenty-nine members of the Association: O. F. H. Bert, J. O. Blumberg, R. C. Briant, Elizabeth F. Brown, A. M. Bryson, W. E. Buker, Helen Calkins, W. E. Cleland, H. B. Curry, L. L. Dines, J. D. Donaldson, H. L. Dorwart, F. A. Foraker, Beatrice L. Hagen, H. C. Hicks, M. L. Manning, L. T. Moston, David Moskovitz, J. H. Neelley, E. G. Olds, F. W. Owens, J. B. Rosenbach, E. M. Starr, R. G. Sturm, E. A. Saibel, J. S. Taylor, W. J. Wagner, E. A. Whitman, E. D. Wells.

The invitation of Pennsylvania State College, to hold the next meeting of the Section at State College, Pennsylvania, was accepted. This meeting is to be held on a Saturday during the month of October, 1942, the exact date to be determined by the Executive Committee.

After an address of welcome by Dean S. C. Crawford of the University of Pittsburgh, the following papers were read:

1. "A discussion of electric tabulating equipment as an aid to statistical work" by P. M. Walter, Jones and Laughlin Steel Corporation, introduced by the Secretary.

2. "Application of Lie theory to differential equations," by Dr. R. F. Clippinger, Carnegie Institute of Technology, introduced by Professor Dines.

3. "Continuity in structural members," by Marshall Holt, Aluminum Research Laboratories, introduced by Dr. Sturm.

4. "Exact values of the trigonometric ratios of certain angles," by Professor H. L. Dorwart, Washington and Jefferson College.

5. "Nomograms for tidal corrections of gravity," by T. A. Elkins, Gulf Research Laboratories, introduced by the Secretary.

6. "Clyde Shepherd Atchison, Ph.D., 1882–1941," by Professor O. F. H. Bert, Washington and Jefferson College.

7. "Symposium: The place of mathematics in the total war effort."

Abstracts of the papers follow:

1. Mr. Walter described a typical installation of electrically operated tabulating equipment consisting of a card punch, collator, horizontal sorter, reproducing punch, and a tabulating machine, and gave a number of examples of the use of this equipment in industrial work.

2. The purpose of Dr. Clippinger's paper was to illustrate for equations of first and second order the following theorem proved by Sophus Lie: If an rth order ordinary differential equation is invariant under the r infinitesimal transformations of an integrable group, its general solution may be found by r quadratures, provided the differential equation is independent of the group in a certain sense.

3. Mr. Holt considered a structural member subjected to direct stress and bending under assumptions which led to the differential equation

$$\frac{d^2y}{dx^2} + \frac{P_y}{EI} = -\frac{M_A}{EI} - \frac{Pe_0}{EI}\sin\frac{\pi x}{L} + \frac{M_B + M_A}{EIL}x.$$

He obtained expressions similar to the ordinary slope-deflection equations for the moments M_A and M_B by a procedure much the same as that outlined by Manderla in which the initial crookedness was not considered.

4. The paper of Professor Dorwart appeared in the Notes and Discussion section of this MONTHLY, May, 1942.

5. Mr. Elkins first defined tidal gravity and explained its value both in theoretical studies of the constitution of the earth and in geophysical prospecting. Nomograms devised by Dr. James B. Friauf for the solution of the astro-

nomical triangle were next discussed. Two types of nomograms for the routine computation of tidal gravity, which can be easily derived from the Friauf nomograms, were then explained.

6. Professor Bert's paper consisted of a biographical sketch of the late Clyde Shepherd Atchison, Ph.D., who had served for nearly thirty years as professor of mathematics at Washington and Jefferson College, together with an estimate of his character and personality as they had impressed themselves upon the author, who had been his colleague for many years. Professor Atchison was shown to have been a man of fine character, lofty ideals, deep convictions, and unswerving devotion. A tireless worker adhering to high standards, he attained great things in his service to his own students and to the cause of education in general. He was pictured as a person of such social graces as to commend him highly as good company, and his passing is widely mourned beyond the bounds of the scholarly and professional fields in which he moved. He was a charter member of the Mathematical Association of America.

7. Dr. Sturm conducted the symposium, and, in his opening remarks, suggested that in addition to the training of other mathematicians, it is possible for mathematicians to contribute to the total war effort by (1) direct service with the armed forces in making the computations involved in the use of ordnance, navigation, and troop movements; (2) the application of mathematical analysis to industrial problems in the manufacture of war materials; and (3) a distinct contribution to the general morale by exercise and stimulation of factual thinking such as is required in mathematics. This latter will be of extreme value in minimizing wishful thinking and susceptibility to propaganda.

Professor H. B. Curry of Pennsylvania State College, a member of the War Preparedness Committee of this Association and of the American Mathematical Society, then reviewed the history and set-up of this committee and urged the members to reread the articles of Chairman Morse and Professor Hart, published in the May and June-July numbers of this MONTHLY for 1941. Professor Curry stated that there had recently been a considerable decrease in the number of graduate students in mathematics so that it is anticipated that there will soon be an acute shortage of mathematicians and mathematics teachers. The shortage on the research level does not seem to be so serious. As the two outstanding needs, Professor Curry cited the training of large numbers of students in elementary mathematics, particularly in trigonometry, and the closing of the gap between pure and applied mathematicians.

Professor J. S. Taylor of the University of Pittsburgh referred to the letter of Admiral Nimitz in the March, 1942, issue of this MONTHLY, and commented on the use of mathematics by the Navy as a yardstick in selecting men for training. While describing courses now being given or in prospect at the University of Pittsburgh, he emphasized that while some college students and graduates have time to take courses on the regular semester basis, others have only one or two months available.

Professor E. G. Olds of Carnegie Institute of Technology stated that although

the present war is often called a mathematician's war, the mathematicians, as such, are not "in the saddle," but are in key positions only when they also happen to have had training in physics, engineering, meteorology, etc. He urged an open-mindedness among teachers of mathematics as regards short courses and emphasis on applications; and urged the formation of an organization to use the spare time of the mathematicians of the Pittsburgh district in industry this summer. Such a committee with Dr. Sturm as chairman was later authorized by the Section during the general discussion which followed these speakers.

H. L. DORWART, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at St. Olaf College, Northfield, Minnesota, on Saturday, May 9, 1942. A morning session, held at 10:30 o'clock, was followed by luncheon and an afternoon session at 2:30 o'clock. Professor E. J. Camp of Macalester College presided at each session, being relieved by the regional governor, Professor Cornelius Gouwens, when giving his own paper.

Seventy persons attended the meeting, including the following thirty-three members of the Association: N. R. Amundson, R. W. Brink, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, S. Elizabeth Carlson, Sister M. Claudette, Paul Cramer, W. S. H. Crawford, Gladys Gibbens, C. H. Gingrich, Cornelius Gouwens, Clara L. Hancock, Dunham Jackson, J. S. Hickman, W. H. Kirchner, W. R. McEwen, W. D. Munro, J. D. Novak, Isaac Opatowski, F. J. Polansky, C. Grace Shover, Abraham Spitzbart, A. J. Strane, L. W. Swanson, F. J. Taylor, H. P. Thielman, Ella Thorp, H. L. Turrittin, A. L. Underhill, K. W. Wegner, Frantisek Wolf, and Sister Thomas à Kempis, institutional member representative.

At the business session officers were elected for the coming year as follows: Chairman: C. H. Gingrich, Carleton College; Secretary: A. L. Underhill, University of Minnesota; Executive Committee: G. C. Priester, University of Minnesota, H. P. Thielman, College of St. Thomas, K. W. Wegner, College of St. Catherine. Professors L. E. Bush and K. W. Wegner were chosen candidates for the position of regional governor for 1943 and 1944.

The following nine papers were presented:

1. "On the separation of parts of a natural boundary of an analytic function" by Dr. Frantisek Wolf, Macalester College.

2. "The discriminant function and its applications" by G. D. Kyle, University of Minnesota, introduced by Professor Underhill.

3. "Mathematical problems arising in the theory of airplane wings" by Professor Albert Gail, University of Minnesota, introduced by Professor Underhill.

4. "Legendre functions of the second kind and related functions" by Professor Dunham Jackson, University of Minnesota.

5. "On a certain functional equation" by Professor H. P. Thielman, College of St. Thomas.

6. "Some instruments for mathematics and physics" by R. M. Sutton, Lecturer in Physics, University of Minnesota, introduced by Professor Bussey.

7. "On the general solution of the nth order linear differential equation" by Professor E. J. Camp, Macalester College.

8. "Square roots from a table of cosines" by W. S. H. Crawford, University of Minnesota.

9. "On the Laplace transformation" by Dr. Isaac Opatowski, University of Minnesota.

Abstracts of these papers follow:

1. Given a function $u(z)$, harmonic in $|z|<1$ and an arc A of $|z|=1$. By B we denote the arc of $|z|=1$ complementary to A. Dr. Wolf gave a method to construct a function $v(z)$, harmonic in $|z|<1$ which is zero continuously on B and such that $u(z)-v(z)$ is zero continuously on A. Given $f(z)$, an analytic function in $|z|<1$, having at least part of $|z|=1$ as a natural boundary, then—applying the above result to the real part of it—he showed that for any A there is a $g(z)$ such that $g(z)$ is analytic on B and $f(z)-g(z)$ is analytic on A. Hence $g(z)$ takes over part of the natural boundary of $f(z)$ and its singular behavior, there, is exactly the same as that of $f(z)$. The proof uses some results from the theory of analytic functions and Parseval's theorem.

2. Mr. Kyle showed that a set of multiple measurements on samples taken from two or more classes may be used to develop a discriminant function, linear in the observations and having the property that better than any other linear function, it will discriminate between any specificed classes. Measurements, p in number, are made upon each object of each sample. Expressions for the difference between the means of the linear function in the specified classes and a quantity proportional to the variance of the linear function within classes are set up, and the ratio of the square of the former to the latter is maximized by variation of each of the unknown coefficients of the linear function independently. This operation leads to a set of p simultaneous linear equations in p unknowns, the unknowns being the desired coefficients of the discriminant function which is readily obtained. Such functions have numerous practical applications in the various branches of science and especially in confident selection, whether in the program of carrying on a war, in industry, or elsewhere.

3. Professor Gail demonstrated Frandtl's equation (for the spanwise circulation distribution of airplane wings) as an example of an integral equation that is commonly used for fundamental calculations in airplane design. He discussed the physical significance of that equation and called attention to its unexplored formal properties.

4. It was shown by Professor Jackson that solutions of the Legendre differential equation for arbitrary positive integral n can be calculated in a wholly elementary manner by means of recurrence relations when the equation has been solved for $n=0$. Alternative choices of a solution for $n=0$ lead to the

Legendre polynomials and to the Legendre functions of the second kind. A similar treatment is applicable to the differential equations of Hermite, Laguerre, Jacobi, and Bessel.

5. Professor Thielman considered the functional equation $f(x+1) = R(x)/F(x)$, where $R(x)$ is any rational function. He gave an explicit solution of this equation in terms of gamma functions, and stated that this was the only monotone and the only either convex or concave solution for large values of x. His derivations were based on, and were extensions of results of earlier articles. (F. John, *Acta Mathematica*, vol. 71, pp. 175–187; H. P. Thielman, *Bulletin of the American Mathematical Society*, vol. 47, pp. 118–120, 1941).

6. Six mechanisms and linkages were demonstrated by Mr. Sutton: (1) a device with spring roller for drawing families of confocal ellipses and hyperbolas, with provision for changing distance between foci; (2) a simple sextant with modified optical system for student use in trigonometry; (3) a combination of plumb-bob with protractor which, as an inclinometer, serves for blackboard use, and for measuring vertical angles in trigonometry and astronomy; (4) a compass using steel tape and rubber suction cup suitable for blackboard use and, when combined with the preceding instrument, for graphing polar coördinate figures; (5) a geometrical linkage based upon the trigonometric law of sines for demonstrating Snell's law of optical refraction; (6) a linkage for graphical solution and visualization of electric circuit problems, showing the interrelation of electromotive force, potential difference, internal resistance, external resistance, current, and power in both internal and external circuits.

7. The usual proof that every solution of the linear differential equation of nth order

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots P_n(x)y = 0 \tag{1}$$

is expressible as a linear combination of n linearly independent solutions, depends on the general existence theorem for linear differential equations. Professor Camp, without making use of the existence theorems of solution of a linear differential equation, proved that if $y_1, y_2, \cdots y_n$ are n linearly independent solutions of equation (1), then every solution is expressible as a linear combination of them. Let $V(x)$ be any solution of equation (1) which is linearly independent of any $n-1$ of the functions $y_1, y_2, \cdots, y_n$. Then Wronski determinants formed from the n combinations of $V(x)$ with $n-1$ of the y_i give a set of linear equations in $V, V^1, \cdots, V^{(n-1)}$. On solving these equations for V, we prove that V is a linear combination of $y_1, \cdots, y_n$ with constant coefficients.

8. Mr. Crawford explained a method whereby the square root of any real number can easily be found by means of a little mental arithmetic and a table of cosines.

9. Some considerations and examples showing the usefulness of the Laplace transformation were given by Dr. Opatowski.

A. L. UNDERHILL, *Secretary*

AN ARITHMETIC FUNCTION ARISING FROM THE ϕ FUNCTION

HAROLD SHAPIRO, Princeton University

1. **Introduction.** In the following discussion we shall consider a rather curious concept which arises very naturally from the iteration of the Euler ϕ function. Since $\phi(x)$ is defined as the number of positive integers, not exceeding x, which are prime to x, it follows immediately that for $x>1$, $\phi(x)<x$. If we write $\phi^2(x)=\phi[\phi(x)]$, $\phi^3(x)=\phi[\phi^2(x)]$, $\cdots$, $\phi^n(x)=\phi[\phi^{n-1}(x)]$, then we have

$$\phi^n(x) < \phi^{n-1}(x), \tag{1}$$

for $\phi^{n-1}(x)>1$. It is well known that for $x=\prod p_i^{\alpha_i}$,

$$\phi(x) = \prod p_i^{\alpha_i-1}(p_i - 1), \tag{2}$$

from which it is easily seen that for $x>2$, $\phi(x)$ is even. Thus we see from (1) that by making n large enough we must always arrive at

$$\phi^n(x) = 2.* \tag{3}$$

When (3) holds we shall say that x is of class n, and write this as $C(x)=n$.

Now it is clear that every positive integer belongs to a uniquely defined class with the exception of 1 and 2. We define $C(1)=C(2)=0$.

It is our purpose, in what follows, to determine several properties of these classes, and properties of the numbers which belong to any one class. We will be able to determine bounds for the function $C(x)$, and to study the general structure of a class.

2. **The fundamental theorem.** First we seek the relationship between $C(xy)$, $C(x)$, and $C(y)$, for any two integers x and y. This is given by

THEOREM 1. *If either x or y is odd*

$$C(xy) = C(x) + C(y) \tag{4}$$

and if both x and y are even

$$C(xy) = C(x) + C(y) + 1. \tag{5}$$

It is obvious that to establish (4) and (5), in general, it is sufficient to prove them for the case when y is a prime. The proof of this will result from the following sequence of theorems.

LEMMA. *For two positive integers x and y, if $(x, y)=1$, then $\phi(xy)=\phi(x)\phi(y)$. On the other hand, if x is a multiple of y, then $\phi(xy)=y\phi(x)$.*

* The function $\phi^n(x)$ was considered by F. de Rocquigny who stated the formula

$$\phi^p(N^m) = \phi^{p-2}(N^{m-2})\phi^{p-1}[(N-1)^2],$$

N a prime, $m>2$, and $p>2\cdots$ L. E. Dickson, "History of the Theory of Numbers," vol. 1, p. 134; F. de Rocquigny, Les Mondes-Revue Hebdom. des Sciences, 48, 1879, 327.

This relation is not true in general as may be seen by taking $N=5$, $m=3$, and $p=6$. Moreover, on the basis of the results of our discussion, we will show that with any prime $N>3$, and $m>2$, the formula does not hold for all p.

The truth of this Lemma follows immediately from (2). Note that for y a prime, the two possibilities considered in the lemma are the only two.

THEOREM 2. *For any odd integer* x, $C(2x) = C(x)$.

This is a consequence of the fact that for x odd, $\phi(2x) = \phi(x)$.

THEOREM 3. *For any even integer* x, $C(2x) = C(x) + 1$.

Proof: Since x is even we have, by the lemma,

$$\phi(2x) = 2\phi(x),$$

and since $\phi(x)$ is even,

$$\phi^2(2x) = 2\phi^2(x).$$

In general, for any $m \leqq C(x)$

$$\phi^m(2x) = 2\phi^m(x),$$

and so taking $m = C(x)$, we have

$$\phi^m(2x) = 2 \cdot 2 = 4,$$

$$\phi^{m+1}(2x) = 2.$$

Thus $C(2x) = m + 1 = C(x) + 1$.

COROLLARY 1. *For* x *odd*, $C(2^a x) = C(2^a) + C(x)$.

COROLLARY 2. *For* x *even*, $C(2^a x) = C(2^a) + C(x) + 1$.

We note here the reason for the *two* conditions, (4) and (5). When we multiply an odd number by 2, the class of the number remains unaltered. On the other hand, for x even, multiplying x by 2 changes its class by 1, and here 2 behaves as if it belonged to class 1, in the sense of (4).

THEOREM 4. *For any integer* x, $C(3x) = C(x) + C(3)$.

Proof: From the lemma we get

$$\phi(3x) = 3\phi(x) \quad \text{or} \quad 2\phi(x),$$

$$\phi^2(3x) = 3\phi^2(x) \quad \text{or} \quad 2\phi^2(x),$$

and in general for any $m \leqq C(x)$, we have

$$\phi^m(3x) = 3\phi^m(x) \quad \text{or} \quad 2\phi^m(x).$$

Then, taking $m = C(x)$ we have

$$\phi^m(3x) = 3 \cdot 2 \quad \text{or} \quad 2 \cdot 2 = 6 \quad \text{or} \quad 4.$$

and

$$\phi^{m+1}(3x) = 2.$$

Hence $C(3x)=m+1=C(x)+1=C(x)+C(3)$.

THEOREM 5. *For any odd prime p, $C(px)=C(p)+C(x)$.*

Proof: The theorem has been shown to be true for the case $p_1=3$. Assuming it true for the first $k-1$ odd primes, $p_1, \cdots, p_{k-1}$, we shall prove it true for the k-th prime, p_k, and thus complete the induction.

From the lemma we have

$$\phi(p_k x) = p_k\phi(x) \quad \text{or} \quad (p_k - 1)\phi(x). \tag{6}$$

Now, if $\phi(p_k x)=(p_k-1)\phi(x)$, we have $(p_k-1)=2^{\alpha}\prod q_i^{\alpha_i}$ where the q_i are odd primes less than p_k, and thus among the first $k-1$. Thus, by Theorem 3, Corollary 2, and our assumption,

$$C[(p_k - 1)\phi(x)] = C(p_k - 1) + C[\phi(x)] + 1.$$

Then,

$$1 + C[\phi(p_k x)] = [C(p_k - 1) + 1] + [C[\phi(x)] + 1]$$

or

$$C(p_k x) = C(p_k) + C(x),$$

which was to be proved.

Thus in (6) there remains to consider the case $\phi(p_k x)=p_k\phi(x)$. In this case we have

$$\phi^2(p_k x) = p_k\phi^2(x) \quad \text{or} \quad (p_k - 1)\phi^2(x).$$

Here the case $\phi^2(p_k x)=(p_k-1)\phi^2(x)$ leads to the required result by an argument similar to that applied to $(p_k-1)\phi(x)$, in (6). Again we are left only to consider

$$\phi^2(p_k x) = p_k\phi^2(x).$$

Continuing this process we see that at each step the only alternative from which our result is not immediate is

$$\phi^m(p_k x) = p_k\phi^m(x).$$

Then taking $m=C(x)$ we get

$$\phi^m(p_k x) = 2p_k,$$

so that $C(p_k x)=m+C(2p_k)=C(x)+C(p_k)$.

As was already mentioned, Theorem 5, and Theorem 3, Corollaries 1 and 2 are sufficient to establish Theorem 1.*

* With regard to de Rocquigny's formula we can now demonstrate its falsity for any $N>3$, $m>2$, and $3<p\leqq C(N^m)$. For if

$$\phi^p(N^m) = \phi^{p-2}(N^{m-2})\phi^{p-1}[(N-1)^2],$$

then the class of the number on the left side of this equation is equal to the class of the number on

Utilizing the relationships (4) and (5) we can readily calculate the integers of the first few classes.*

Class	Numbers of This Class
0	1, 2,
1	3, 4, 6,
2	5, 7, 8, 9, 10, 12, 14, 18,
3	11, 13, 15, 16, 19, 20, 21, 22, 24, 26, 27, 28, 30, 36, 38, 42, 54,
4	17, 23, 25, 29, 31, 32, 33, 34, 35, 37, 39, 40, 43, 44, 45, 46, 48, 49, 50, 52, 56, 57, 58, 60, 62, 63, 66, 70, 72, 74, 76, 78, 81, 84, 86, 90, 98, 108, 114, 126, 162,
5	41, 47, 51, 53, 55, 59, 61, 64, 65, 67, 68, 69, 71, 73, 75, 77, 79, 80, 82, 87, 88, 91, 92, 93, 94, 95, 96, 99, 100, 102, 104, 105, 106, 109, 110, 111, 112, 116, 117, 118, 120, 122, 124, 127, 129, 130, 132, 133, 134, 135, 138, 140, 142, 144, 146, 147, 148, 150, 152, 154, 156, 158, 163, 168, 171, 172, 174, 180, 182, 186, 189, 190, 196, 198, 210, 216, 218, 222, 228, 234, 243, 252, 254, 258, 266, 270, 294, 324, 326, 342, 378, 486,
6	83, 85, 89, 97, $\cdots$
7	137, $\cdots$
8	257, $\cdots$

Brief as this table may be it furnishes us with an excellent basis for conjecture and inductive proof.

3. **Bounds for C(x).** Noting that the table above gives *all* the numbers of each of the first few classes, we now proceed to determine bounds for $C(x)$.

THEOREM 6. *The largest odd number in the class m is 3^m, and the largest even number in the class m is $2 \cdot 3^m$.*

Proof: From the table above we see that the theorem is true for $m = 0, 1, 2, 3$. Now we assume it to be true for all $m' < m$, and will prove it true for class m, thus completing the proof of the theorem.

First, for any odd prime p in class m we have $C(p-1) = m-1 < m$, and therefore, by our assumption, $p-1 \leqq 2 \cdot 3^{m-1}$. Then $p \leqq 2 \cdot 3^{m-1}+1 \leqq 3^m$.

Next, for an odd prime p, if p^α, $\alpha > 1$, is in class m, we have $C(p^\alpha) = \alpha C(p) = m$, and $C(p) = m/\alpha < m$. Hence, by our assumption $p \leqq 3^{m/\alpha}$ and $p^\alpha \leqq 3^m$.

---

the right. This implies that either

(1) $mC(N) - p = (m-2)C(N) - (p-2) + 2C(N-1) + 1 - (p-1) + 1 = mC(N) - 2p+3$,

or

(2) $mC(N) - p = (m-2)C(N) - (p-2)$,

or

(3) $mC(N) - p = 2C(N-1) - p + 2 = 2C(N) - p$,

which give respectively that we must have either (1) $p=3$, (2) $C(N)=1$, which for prime N is true only if $N=3$, or (3) $m=2$.

* The completeness of this table, as given, can be verified from the tables of solutions of the equation $\phi(x)=n$, $\cdots$ Number-Divisor Tables, Br. Ass. Mathematical Tables, V. 8, $\cdots$ or by utilizing known methods of solving this equation as given by Wright, Theory of Numbers, and Carmichael, Amer. Journal of Math., 30, 1908, p. 394–400.

Now, if $n = 2^{\alpha}\prod p_i^{\alpha_i}$* is any composite number such that $C(n) = m$, we know from Theorem 1 that

$$\left.\begin{aligned} C(n) &= \sum C(p_i^{\alpha_i}), \quad \text{for} \quad \alpha = 0; \\ &= \sum C(p_i^{\alpha_i}) + \alpha - 1, \quad \text{for} \quad \alpha \geqq 1. \end{aligned}\right\}$$

Then, since for each of these $p_i^{\alpha_i}$, $p_i^{\alpha_i} \leqq 3^{C(p_i^{\alpha_i})}$, $C(p_i^{\alpha_i}) \geqq \dfrac{\log p_i^{\alpha_i}}{\log 3}$ and we get

$$\left.\begin{aligned} C(n) &\geqq \sum \frac{\log p_i^{\alpha_i}}{\log 3}, \quad \text{for} \quad \alpha = 0; \\ &\geqq \sum \frac{\log p_i^{\alpha_i}}{\log 3} + \alpha - 1, \quad \text{for} \quad \alpha \geqq 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\geqq \frac{\log n}{\log 3}, \quad \text{for} \quad \alpha = 0; \\ &\geqq \frac{\log n}{\log 3} - \frac{\log 2}{\log 3}, \quad \alpha \geqq 1. \end{aligned}\right\}$$

From this we see that for n odd, $n \leqq 3^{C(n)}$, and for n even, $n \leqq 2 \cdot 3^{C(n)}$. This completes the induction and the proof of our theorem.

COROLLARY: *The maximum value of x for which $C(x) = m$ is $2 \cdot 3^m$. In other words $x \leqq 2 \cdot 3^{C(x)}$.*

THEOREM 7. *The smallest even number of class m is 2^{m+1}.*

Proof: We note from our table that the theorem is true for $m = 0, 1, 2, 3$, and we now assume it true for all classes $m' < m$. Suppose next that $C(s) = m$, where $s = 2^{\alpha}r$, r odd. Now if $\alpha > 1$, $s = 2 \cdot 2^{\alpha-1}r$, $\alpha - 1 > 0$, and $C(2^{\alpha-1}r) = m - 1$. Then $2^{\alpha-1}r \geqq 2^m$, by our assumption, and hence $s = 2^{\alpha}r \geqq 2^{m+1}$.

On the other hand, if $\alpha = 1$, $s = 2r$, and it remains to show $2r \geqq 2^{m+1}$. Suppose $2r < 2^{m+1}$; then $r < 2^m$, where r is odd, and $C(r) = m$. Thus $\phi(r) < r < 2^m$, but $\phi(r)$ is even and $C[\phi(r)] = m - 1$, so that by our assumption $\phi(r) \geqq 2^m$. From this contradiction we see that $2r \geqq 2^{m+1}$, and our proof is completed.

THEOREM 8. *The smallest odd number of class m is greater than 2^m.*

Proof: Let s be the smallest odd number of class m. Then, by Theorem 7, $2s > 2^{m+1}$, and $s > 2^m$.

COROLLARY. *The smallest number in class m is greater than 2^m. In other words $x > 2^{C(x)}$.*

From the corollaries to Theorems 6, and 8, we have

$$2^{C(x)} < x \leqq 2 \cdot 3^{C(x)}$$

* This represents the unique factorization of n as a product of powers of distinct primes.

and the bounds for $C(x)$ are

$$\frac{\log x}{\log 2} > C(x) \geqq \frac{\log x/2}{\log 3}.$$

4. A lower bound for $\phi(x)$. In passing, it is interesting to note that we can derive a lower bound for $\phi(x)$, by utilizing the properties of classes which we have developed thus far. From Theorem 7, since $\phi(x)$ is even, $\phi(x) \geqq 2^{C(x)}$. Also, we know from Theorem 6 that for x odd, $x \leqq 3^{C(x)}$ or $C(x) \geqq \log x/\log 3$. Thus combining these two facts we get

$$\phi(x) \geqq 2^{C(x)} \geqq 2^{\log x/\log 3} = x^{\log 2/\log 3}, \tag{7}$$

for all odd x. Having obtained the inequality (7) for all odd x, the question naturally arises as to whether or not it holds also for even x. In this connection we shall digress slightly and prove that (1) The only solutions of the equation $\phi(x) = x^{\log 2/\log 3}$ are $x = 1$, and 3; and (2) For all integers x, except $x = 1, 2, 3, 4, 6, 10, 12, 18$, and 30, $\phi(x) > x^{\log 2/\log 3}$.

LEMMA 1. *If $M = M(x)$ stands for the number of distinct prime divisors of the integer x, then for all odd x, with the exception of $x = 1, 3, 5, 7, 9, 15$, and 21,*

$$\log x > \frac{(\log 2)^2}{\log 3/2} + M \log 3. \tag{8}$$

Proof: For all odd $x > 5$, which do not contain a factor 3, it is obvious that

$$x > 5^M. \tag{9}$$

Now for those odd $x > 5$, which are divisible by 3, (9) holds, except for $x = 15$, and 105. Thus from (9) we have that for x odd and greater than 5, $x \neq 15, 105$, $\log x > M \log 5$.

Next we show that

$$M \log 5 > \frac{(\log 2)^2}{\log 3/2} + M \log 3, \tag{10}$$

for $M \geqq 3$. This follows readily since (10) simplifies to

$$M > \frac{(\log 2)^2}{(\log 3/2)(\log 5/3)} = 2.3 \cdots .$$

Now for $M = 1, 2$, the only odd x not satisfying (8) are $x = 1, 3, 5, 7, 9, 15$, and 21. Also, $x = 105$ satisfies (8), so that the proof of the Lemma is completed.

LEMMA 2. *For all odd x, with the exception of $x = 1, 3, 5, 7, 9, 15$, and 21.*

$$x(2/3)^M > (2x)^{\log 2/\log 3}. \tag{11}$$

Proof: This is immediate since (8) may be reduced to (11).

From Lemma 2 we get immediately

$$\phi(x) \geqq x(2/3)^M$$

$$> (2x)^{\log 2/\log 3} \tag{12}$$

$$> x^{\log 2/\log 3} \tag{13}$$

for all odd $x \neq 1, 3, 5, 7, 9, 15, 21$. However, actual trial shows that of these seven integers 1 and 3 are the only ones for which $\phi(x) \geqslant x^{\log 2/\log 3}$. For $x=1$, and 3, $\phi(x) = x^{\log 2/\log 3}$, so that 1 and 3 are the only odd solutions of this last equation.

Furthermore $x=7$ and 21 also satisfy (12) so that the only exceptions there are $x=1, 3, 5, 9$, and 15.

Next we consider the even numbers which are of the form $x' = 2^\alpha x$, $\alpha > 0$, x odd and $\neq 1, 3, 5, 9$, or 15. Then

$$\begin{aligned}\phi(x') &= 2^{\alpha-1}\phi(x) > 2^{\alpha-1} \cdot 2^{\log 2/\log 3} \cdot x^{\log 2/\log 3} \\ &= 2^{\alpha-1+\log 2/\log 3} \cdot (x'/2^\alpha)^{\log 2/\log 3} \\ &= 2^{(\alpha-1)(1-\log 2/\log 3)} \cdot (x')^{\log 2/\log 3} \\ &\geqq (x')^{\log 2/\log 3},\end{aligned}$$

so that

$$\phi(x') > (x')^{\log 2/\log 3}.$$

There remains to be considered the cases

$$x' = 2^\alpha y, \qquad y = 1, 3, 5, 9, \text{ and } 15.$$

Here $\phi(x') > (x')^{\log 2/\log 3}$ implies

$$2^{\alpha-1}\phi(y) > (2^\alpha y)^{\log 2/\log 3}$$

$$(\alpha - 1)(\log 2)(\log 3) + (\log 3)(\log \phi(y)) > \alpha(\log 2)^2 + (\log 2)(\log y)$$

$$\alpha > \frac{(\log 2)(\log 3) + (\log 2)(\log y) - (\log 3)(\log \phi(y))}{(\log 2)(\log 3) - (\log 2)^2}, \tag{14}$$

so we see that for a given y, when α is large enough our result must certainly follow. For $y=1, 3, 5, 9$, and 15, (14) gives that $\alpha \geqq 3, 3, 2, 2, 2$, respectively, is necessary and sufficient, so that the only x for which (13) does not hold are $x=1, 2, 3, 4, 6, 10, 12, 18$, and 30.

Also, since none of the even numbers among these satisfies $\phi(x) = x^{\log 2/\log 3}$, $x=1$, and 3, are the only solutions.

5. The structure of a class. We note from our discussion thus far that each class has, in a sense, a definite structure. We know that in class m we will always find the numbers 2^{m+1}, 3^m, and $2 \cdot 3^m$. This plus the important role they play in each class, as indicated by the previous theorems, suggests that we divide class m into sections, as follows:—

$$(m): \quad 2^m < \underbrace{s, \cdots,}_{\text{I}} \quad \underbrace{2^{m+1}, \cdots, 3^m,}_{\text{II}} \quad \underbrace{\cdots, 2 \cdot 3^m}_{\text{III}}$$

From the theorems already proved we know immediately that the numbers of section

$$\begin{cases} \text{I are all odd.} \\ \text{II may be either odd or even.} \\ \text{III are all even.} \end{cases}$$

We now proceed to deduce several other properties peculiar to the integers lying in the same section of their class.

If we let $x = 2^\alpha s$, s odd and >1, $\alpha \geqq 1$, and $C(x) = m$, we have $C(2^\alpha s) = C(2^\alpha) + C(s) = \alpha - 1 + C(s) = m$. We know that

$$s > 2^{C(s)}$$

and thus,

$$2^\alpha s > 2^{\alpha + C(s)} = 2^{m+1}. \tag{15}$$

Also, since s is odd,

$$s \leqq 3^{C(s)}$$

$$2^\alpha s \leqq 2^\alpha 3^{C(s)} < 3^{\alpha - 1} 3^{C(s)}, \qquad \alpha \geqq 3,$$

$$2^\alpha s < 3^{\alpha - 1 + C(s)} = 3^m. \tag{16}$$

From (15) and (16) and the already mentioned fact that the numbers in section III of any class are even, we have the following theorems.

THEOREM 9. *For any integer x in section III of its class, $x \equiv 0 \pmod 2$, but $x \not\equiv 0 \pmod 8$.*

COROLLARY. *If $\log x / \log 3 > C(x)$, then $x \equiv 0 \pmod 2$, and $x \not\equiv 0 \pmod 8$.*

Theorem 9, though it is a simple result, gives us, approximately, the structure of section III of any class m. This section includes those integers formed as follows:

(a) 2 times any odd number $s > 3^{m/2}$, for which $C(s) = m$.

(b) 4 times any odd number $s > 3^m/4$, for which $C(s) = m - 1$.

However, we may go farther, and give a more complete determination of the integers in section III of any class, by giving explicitly all the integers which come under (b), and by finding narrower bounds for the s which need be considered under (a). In order to do this we need to consider certain portions of section II of a class.

THEOREM 10. *There exist no odd x such that*

$$3^{m-2} \cdot 7 < x < 3^m, \tag{17}$$

where $C(x) = m$.

Proof: We note from the table that the theorem is true for $m=0, 1, 2, 3$, and proceeding by induction, assume it true for all $m'<m$, $m\geqq 3$. Now let us suppose that for an odd integer x, $C(x)=m$, and we have

$$3^{m-2}\cdot 7 < x < 3^m. \tag{18}$$

If x is a prime, $C(x-1)=m-1$, and $x-1\leqq 2\cdot 3^{m-1}$, but $x-1>3^{m-2}\cdot 7-1>2\cdot 3^{m-1}$ for $m\geqq 3$. Thus x cannot be a prime, and it certainly is not of the form 3^a, so that we may write $x=st$, $3^{C(s)}>s>1$, $3^{C(t)}\geqq t>1$. Then from (18) we get

$$3^{C(s)} > s > 3^{C(s)+C(t)-2}\cdot 7/t \geqq 3^{C(s)-2}\cdot 7. \tag{19}$$

But $C(s)<m$, and s is odd, so that (19) contradicts our original assumption. Hence, the assumption that an odd x satisfying (17) exists, is false, and our induction is complete.

THEOREM 11. *There exists no odd x such that*

$$\cdots \qquad 3^{m-3}\cdot 19 < x < 3^{m-2}\cdot 7, \tag{20}$$

where $C(x)=m$.

Proof: Proceeding by induction as in the proof of Theorem 10 we note that the theorem is true for $m=0, 1, 2, 3$, and assume it true for all $m'<m$, $m>3$. Again we suppose the existence of an odd integer x of class m such that (20) holds. As in Theorem 10, x cannot be a prime since then $x-1\leqq 2\cdot 3^{m-1}$, and at the same time $x-1>3^{m-3}\cdot 19-1>2\cdot 3^{m-1}$, for $m>3$. Thus since x is composite and not of the form 3^a, we may write $x=st$, $s>1$, $t>1$, where, by Theorem 10, we may assume $s\geqq 3^{C(s)-2}\cdot 7$. Now we cannot have $s=3^{C(s)-2}\cdot 7$, and $t=3^{C(t)}$, for then $x=st=3^{C(s)+C(t)-2}\cdot 7=3^{C(x)-2}\cdot 7$. Hence, either $s<3^{C(s)-2}\cdot 7$, or $t\leqq 3^{C(t)-2}\cdot 7$. Also, we cannot have $s=7\cdot 3^{C(s)-2}$ and $t=7\cdot 3^{C(t)-2}$, for then $x=st=49\cdot 3^{C(x)-4}=3^{m-4}\cdot 49<3^{m-3}\cdot 19$. Thus we may assume $s<3^{C(s)-2}\cdot 7$, and from (20) we get

$$3^{C(s)-2}\cdot 7 > s > 3^{C(s)+C(t)-3}\cdot 19/t > 3^{C(s)-3}\cdot 19. \tag{21}$$

But $C(s)<m$, and s is odd, so that (21) contradicts our original assumption. Hence the assumption that an odd x satisfying (20) exists, is false, and the proof of the theorem is complete.

COROLLARY. *The three largest odd numbers of class m, $m\geqq 3$, are in order of magnitude, $19\cdot 3^{m-3}$, $7\cdot 3^{m-2}$, and 3^m.*

THEOREM 12. *The numbers of section II of class m are*

(a) $2\cdot 3^{m-3}\cdot 19$, $2\cdot 3^{m-2}\cdot 7$, $2\cdot 3^m$; *and 2 times any odd number s, $C(s)=m$, such that $3^m/2<s<19\cdot 3^{m-3}$;*

(b) $4\cdot 3^{m-3}\cdot 7$, *and* $4\cdot 3^{m-1}$.

Proof: From the discussion immediately following Theorem 9, we already have the integers of section III of class m separated into two sets (a) and (b).

In (a) we have 2 times any odd number $s>3^m/2$ for which $C(s)=m$. Thus by the Corollary to Theorem 11 we have that these numbers are $2\cdot 3^{m-3}\cdot 19$, $2\cdot 3^{m-2}\cdot 7$, $2\cdot 3^m$; and 2 times any odd number s, $C(s)=m$, such that $3^m/2<s<19\cdot 3^{m-3}$. Since $3^m/4>19\cdot 3^{m-4}$, the only odd s fitting the description under (b), on page 25, are $7\cdot 3^{m-3}$, and 3^{m-1}. Thus the only numbers of section III of class m which are divisible by 4, are the two given in the theorem.

In addition to giving us this more complete determination of the integers of section III of any class, Theorems 10, and 11 reflect upon the prime numbers which can occur in certain parts of section II of any class. We will see that restricting a prime number to certain parts of its class, limits its form very definitely.

THEOREM 13. *A prime number p ($\neq 5$) satisfies the inequality*

$$p > 43\cdot 3^{C(p)-4}, \tag{22}$$

if and only if $p=2\cdot 3^{C(p)-1}+1$.

Proof: For $p=2\cdot 3^{C(p)-1}+1$, it is obvious that p satisfies (22). On the other hand, if any prime p satisfies (22), then it remains for us to prove that it must equal $2\cdot 3^{C(p)-1}+1$.

We will assume $C(p)>4$, which is valid since an inspection of the table shows the theorem to be true for the first four classes, (excluding the class 0). Then

$$p-1>42\cdot 3^{C(p)-4}>3^{C(p)-1}=3^{C(p-1)}$$

so that $p-1$ is in section III of its class. Hence $p-1=2x'$, or $4x'$, x' odd. If $p-1=4x'$, then

$$x'=(p-1)/4>(42/4)\cdot 3^{C(p)-4}=(42/36)\cdot 3^{C(p)-2}>3^{C(x')},$$

which is impossible. On the other hand, if $p-1=2x'$,

$$x'=(p-1)/2>(42/2)\cdot 3^{C(p)-4}=(42/6)\cdot 3^{C(p)-3}=7\cdot 3^{C(p)-3},$$

so that by Theorem 10, $x'=3^{C(p)-1}$, and $p=2\cdot 3^{C(p)-1}+1$.

THEOREM 14. *A prime number p satisfies the inequalities*

$$43\cdot 3^{C(p)-4}>p>39\cdot 3^{C(p)-4} \tag{23}$$

if and only if $p=2\cdot 7\cdot 3^{C(p)-3}+1$.

Proof: Here again we may assume $C(p)>4$. Also, since a prime of the given form obviously satisfies (23), it remains only to prove the converse. If a prime p satisfies (23), then

$$p-1>39\cdot 3^{C(p)-4}-1>3^{C(p)-1}=3^{C(p-1)}$$

so that $p-1$ is in section III of its class. Thus $p-1=2x'$, or $4x'$, x' odd. If $p-1=4x'$, $x'=p-1/4>(38/4)\cdot 3^{C(p)-4}>3^{C(p)-2}$, which is impossible. For

$p-1=2x'$, $x'=(p-1)/2>(38/2)\cdot 3^{C(p)-4}=19\cdot 3^{C(p)-4}$ so that by Theorems 10, and 11, $x'=7\cdot 3^{C(p)-3}$, or $3^{C(p)-1}$. However, if $x'=3^{C(p-1)}$, then we would have $p>43\cdot 3^{C(p)-4}$, so that we must have $x'=7\cdot 3^{C(p)-3}$, or $p=2\cdot 7\cdot 3^{C(p)-3}+1$.

6. Section I of a class. We next consider some properties of the integers of section I of any class.

THEOREM 15. *If an integer x is in section I of the class to which it belongs, then every divisor of x is in section I of its class.*

Proof: Since x is in section I of its class, we know that x is odd. Now, our theorem is obviously true for x an odd prime, so that we need only consider the case where x is composite. Then, if $x>d>1$, where $x=ds$, we have

$$2^{C(x)+1} = 2^{C(ds)+1} > x > 2^{C(ds)} = 2^{C(x)}$$

and $ds<2^{C(ds)+1}=2^{C(d)+C(s)+1}$, since both d and s are odd. If $d>2^{C(d)+1}$, *i.e.* if d were not in section I of its class, since $s>2^{C(s)}$, we would have $ds>2^{C(d)+C(s)+1}$ which is a contradiction. From this we have the truth of our theorem.

It is now reasonable to ask whether the converse of Theorem 15 is true; *i.e.* if all the proper divisors of a number x are in section I of their respective classes, is x in section I of its class? The answer is obviously, no, since if we consider any two odd numbers s and t such that

$$\begin{aligned} 2^{C(s)}\sqrt{2} < s < 2^{C(s)+1} \\ 2^{C(t)}\sqrt{2} < t < 2^{C(t)+1}. \end{aligned} \tag{24}$$

Then $st>2^{C(s)+C(t)+1}=2^{C(st)+1}$, and thus st is in section II of its class. That numbers s and t satisfying (24) actually exist is easily verified.

In connection with the existence of numbers for which every proper divisor is in section I of its class whereas the number itself is in section II of its class, we have the following interesting theorem.

THEOREM 16. *If P is any number of the form 2^n-1 which is in section I of its class, and s is any number also in section I of its class; then, if $C(s)\leqq C(P)$, $x=sP$ is in section II of its class.*

Proof: From our hypothesis

$$2^{C(s)} < s < 2^{C(s)+1} \tag{25}$$

and

$$P = 2^{C(P)+1} - 1. \tag{26}$$

Also,

$$C(s) \leqq C(P) < C(P) + 1,$$

$$2^{C(P)+1} - 1 > 2^{C(s)},$$

$$2^{C(s)+C(P)+1} + 2^{C(P)+1} - 1 - 2^{C(s)} > 2^{C(s)+C(P)+1},$$

$$(2^{C(s)} + 1)(2^{C(P)+1} - 1) > 2^{C(sP)+1}.$$

Then from (25)

$$s(2^{C(P)+1}-1) \geqq (2^{C(s)}+1)(2^{C(P)+1}-1) > 2^{C(sP)+1},$$

and combining this with (26) we get

$$sP > 2^{C(sP)+1},$$

so that sP is in section II of its class.

COROLLARY. *If x is any number in section I of its class, then*

$$x \not\equiv 0 \pmod{P^2}, \tag{27}$$

where P is any number of the form 2^n-1.

Proof: If P is not in section I of its class, then we have by Theorem 15, $x \not\equiv 0 \pmod{P}$ which certainly implies (27). If P is in section I of its class, letting $s=P$, we see that the conditions of Theorem 16 are satisfied, and thus P^2 is in section II of its class. Then again by Theorem 15, $x \not\equiv 0 \pmod{P^2}$, which completes the proof of the corollary.

We note that in discussing numbers of the form $P=2^n-1$ which are in section I of their class, Theorem 16 and its corollary also include the special case where P is a Mersenne prime. However, with respect to Mersenne primes the hypothesis that they be in section I of their class is a strong limitation. In fact, in the following we determine all such primes.

LEMMA. *$C(2^x+1)=x$ if and only if 2^x+1 is a Fermat prime.*

Proof: If 2^x+1 is a prime, then obviously $C(2^x+1)=x$. Now, if 2^x+1 is not a prime, $\phi(2^x+1)<2^x$, and since $\phi(2^x+1)$ is even, by Theorem 7 we see that $C[\phi(2^x+1)] \leqq x-2$, and hence $C(2^x+1)=1+C[\phi(2^x+1)] \leqq x-1<x$.

THEOREM 17. *The only Mersenne primes which are in section I of their class are 2^2-1, 2^3-1, 2^5-1, and $2^{17}-1$.*

Proof: If a Mersenne prime $M_p=2^p-1$ is in section I of its class we have

$$C(2^p-1) = p-1$$

$$\begin{aligned} C(2^p-1) &= 1+C(2^{p-1}-1) \\ &= 1+C(2^{(p-1)/2}-1)+C(2^{(p-1)/2}+1), \quad p \geqq 3. \end{aligned}$$

But

$$C(2^{(p-1)/2}-1) \leqq (p-1)/2-1$$

$$C(2^{(p-1)/2}+1) \leqq (p-1)/2,$$

and thus

$$1+C(2^{(p-1)/2}-1)+C(2^{(p-1)/2}+1) \leqq p-1. \tag{28}$$

However, the equality sign holds in (28) so that

$$C(2^{(p-1)/2}+1) = (p-1)/2 \tag{29}$$

$$C(2^{(p-1)/2}-1) = (p-1)/2 - 1. \tag{30}$$

By the lemma, and (29) we see that $2^{(p-1)/2}+1$ is a Fermat prime and hence

$$\frac{p-1}{2} = 2^m \tag{31}$$

$$p = 2^{m+1}+1. \tag{32}$$

Combining (30) and (31) we get

$$C(2^{2^m}-1) = 2^m - 1.$$

Then assuming $m>0$, we have

$$C(2^{2^{m-1}}-1) + C(2^{2^{m-1}}+1) = 2^m - 1.$$

But

$$C(2^{2^{m-1}}-1) \leqq 2^{m-1}-1$$
$$C(2^{2^{m-1}}+1) \leqq 2^{m-1},$$

so that

$$C(2^{2^{m-1}}-1) + C(2^{2^{m-1}}+1) \leqq 2^m - 1. \tag{33}$$

Since the equality sign holds in (33) we have

$$C(2^{2^{m-1}}-1) = 2^{m-1}-1 \tag{34}$$

$$C(2^{2^{m-1}}+1) = 2^{m-1}. \tag{35}$$

Then by the lemma, and (35) we see that $2^{2^{m-1}}+1$ is also a Fermat prime. By continuing this process we can then show that $2^{2^{m-2}}+1$ must also be a Fermat prime, and so on. From this, and the fact that $2^{2^5}+1$ is not a prime, we have $m \leqq 4$. Then from (32), and the fact that we originally assumed $p \geqq 3$, we see that p may be either 2, 3, 5, 17. Then our Mersenne primes may be correspondingly 2^2-1, 2^3-1, 2^5-1, $2^{17}-1$. That these are all primes is known, and that they are all in section I of their respective classes is easily verified.

From our discussion we have been able to get some idea of the constitution of a class. But this discussion is by no means complete, and there exist numerous conjectures which we have been unable to resolve one way or another. Foremost among these conjectures is the proposition, justified by calculation with respect to the first eight classes, that the smallest number in each class is a prime. This may prove to be very difficult to establish. So far we have been unable even to demonstrate that there is at least one prime in each class.

ADJUSTMENTS IN MATHEMATICS TO THE IMPACT OF WAR

G. B. PRICE, University of Kansas

Frank B. Jewett said in a public address [1] early in 1942: "Without insinuating anything as to guilt, the chemists declare that this is a physicist's war. With about equal justice one might say that it is a mathematician's war." This note contains a report on how the mathematicians have participated in the war, the problems that have arisen, the measures that have been taken to meet them, the changes that have been made in the mathematics curriculum, and the effects that war has had on graduate work in mathematics [2]. It is based on information obtained through questionnaires [3] from forty-one of the fifty-four institutions in the United States and Canada that offer the Ph.D. degree in mathematics and from thirteen additional colleges and universities in the United States. No information was obtained from Canada. It is important to observe that the information was gathered between November 17 and December 15, 1942. Since the government has now announced (December 17, 1942) its program for the utilization of the colleges and universities during the war, this report can be considered a record of the first year, and of the first phase, of the participation of mathematics in the war.

The report can be summarized in the following conclusions.

1. There has been an enormous increase in enrollments in mathematics courses; although the most frequently reported increase was thirty per cent (enrollment for the first semester of 1942–43 compared with enrollment for the first semester of 1941–42), many institutions reported increases up to sixty and seventy per cent, and six colleges and universities reported increases ranging from 100 per cent to 300 per cent. These increases took place in spite of decreased university enrollments and enlarged summer schools during the past summer. The number of men studying mathematics is large; for example, in the first semester of 1942–43 at the University of Michigan 939 men were enrolled in undergraduate and first-year graduate courses in mathematics in the arts college and 1820 others were in similar courses in the engineering college.

2. The percentage increase in enrollments in mathematics is much smaller for women than for men. Four women's colleges in the East reported increases ranging from twenty-five to forty per cent. Most coeducational institutions reported either no increase or only a small one; exceptions are one university, 62 per cent; three colleges and universities, 100 per cent; one university, 125 per cent. The total number of women studying mathematics is small. For example, the University of Michigan, after a ten per cent increase over last year, had only 337 women in undergraduate and first-year graduate courses in mathematics in the arts college; approximately two-thirds of this group is enrolled in first-year courses. Michigan apparently has one of the largest groups of women studying mathematics [4]. Of two large universities reporting 100 per cent increases in women in mathematics, one still has fewer than 150 in all undergraduate and graduate courses and the other has fewer than 200.

3. There exists a tremendous and acute shortage of mathematics instructors.

This shortage is attested by such statements as "we could have used four additional instructors this fall" and "twenty per cent of the staff is in the army or navy," but perhaps even more by the emergency steps taken to secure instructors. These include (1) increasing the teaching load of the individual instructor; (2) increasing the size of sections; (3) offering only essential courses (this step has involved dropping many undergraduate and graduate courses); (4) abolishing summer vacations (this step involves continuous operation under a three semester or four quarter plan); (5) using practically all graduate students as instructors and even some senior undergraduates as instructors, assistants, and readers; (6) calling retired professors back to active duty, engaging former mathematics teachers who had entered business or other occupations, and employing retired Army officers; (7) borrowing instructors from other departments (in some cases they are given refresher or preparatory courses [5]); borrowing instructors for part-time work from a local air base or from some government agency; (8) "robbing other institutions"; (9) employing faculty wives.

4. Spherical trigonometry, in many cases combined with solid geometry, and the mathematics of navigation have been added to the curriculum almost universally.

5. Ten schools reported that they had added additional elementary courses, other than solid geometry and spherical trigonometry, in plane geometry, algebra, and trigonometry, or some combination of these, to enable students to remove high school deficiencies in mathematics or to obtain training needed for entrance into some branch of the armed forces. Some of this work, especially in algebra and trigonometry, has always been given in many institutions.

6. There has been practically no change in the content or organization of the foundation courses in mathematics (algebra, trigonometry, analytic geometry, and calculus); in continuing to teach the fundamentals in courses which have long been standard the mathematicians have carried out the recommendations of the Army, the Navy, and of their own War Preparedness Committee.

7. Fifteen institutions reported that they had dropped some of their undergraduate courses, especially advanced courses in pure mathematics but also mathematics courses for special groups of students. Among the courses not being given or being given with less frequency are pure geometry, number theory, history of mathematics, fundamental concepts of mathematics, and survey courses in mathematics; forestry mathematics, and various courses for students of business. The reasons for dropping these courses are (1) students are taking standard courses instead of special ones; (2) students are taking courses in applied mathematics instead of in pure and abstract mathematics; (3) the shortage of instructors has made it necessary to drop courses in order to obtain instructors for elementary courses. The shift in the curriculum indicated here and in the next conclusion has already assumed large proportions.

8. There has been a big increase in teaching and research in applied mathematics: fifty-seven *new courses* (other than those named in 4 and 5 above) in approximately twenty different subjects in applied mathematics are being given

in the fifty-four institutions reporting. There are eight courses in mathematics of artillery fire or exterior ballistics; eight courses in cryptography and cryptanalysis; five courses in principles of mechanics; five courses in mathematics for meteorology; and many others with smaller frequencies. In addition, there was inaugurated at Brown University in the summer of 1941 a notable program of Advanced Instruction and Research in Mechanics; Brown will begin publication of *Journal of Applied Mathematics* early in 1943. The University of Wisconsin has established a four-year course leading to the degree of Bachelor in Applied Mathematics and Mechanics.

9. Mathematicians are doing a large amount of teaching in training schools for the Army and Navy, in ESMWT courses, and in refresher courses for those about to enter the army and navy. These schools instruct in various branches of engineering, meteorology, radio and electronics, Diesel motors, navigation and other pre-flight subjects, and train workers for aircraft plants, research workers for aeronautics laboratories, mathematics teachers for the Army and Navy, and so on. The information gathered on this subject is incomplete because the mathematics in these schools is often under the administrative control of the engineering school or some other department of the university. Two examples will illustrate the proportions to which this work has grown. The mathematics department in a state school in the mid-west, far from water of any kind, is giving instruction in mathematics continuously to 1600 students in two training schools for the Navy. The Department of Mathematics of the University of California at Los Angeles is giving approximately forty courses for aircraft workers; during the summer it conducted a series of courses in anti-aircraft mathematics for the men enlisted in the Coast Artillery assigned to anti-aircraft stations in California [6].

10. Five institutions have provided special courses involving mathematics for the training of women; in general, however, there have been no special provisions for women and no concerted efforts to enlist them in the study of mathematics. Purdue is introducing a new sequence of courses to prepare women as statistical workers. Chicago has a special course in mathematics for women students of electronics. The University of Pennsylvania is providing a special course in engineering drafting which requires mathematics. The University of California at Los Angeles has blocked out several curricula designed to prepare women as aircraft workers [6]; these include plane and solid analytic geometry, engineering mechanics, descriptive geometry, and so on. The University of New Mexico is planning a two-year specialized program for women that will involve the College of Engineering and the Departments of Mathematics and Physics. Smith College has plans under way for a summer school next summer that will likely emphasize mathematics and science. All the information indicates a strong demand for women trained in mathematics and an increasing tendency for them to enter the field. Positions on all levels are open to them. The case is cited of a young woman who graduated from a university in the mid-west with an A.B. degree in mathematics (and a Phi Beta Kappa key!) and who obtained an at-

tractive position in the Research Laboratory of the United Aircraft Corporation at East Hartford, Connecticut.

11. Graduate work in mathematics is rapidly approaching the vanishing point. One school reported that such work had been abolished; in many others it is rapidly disappearing as a result of (a) the loss of graduate students, (b) the employment of graduate instructors on war research and for elementary instruction, and (c) the use of graduate students as instructors. One large university reports that it is offering only ten graduate courses this year instead of the usual fifteen, and that only five are planned for next year. Many institutions reported decreases in enrollment in graduate courses in mathematics ranging from fifty to seventy-five per cent. The following are the actual numbers of graduate students in mathematics reported at ten institutions: Brown, 20; Cincinnati, 5; George Washington, 20; Iowa State, 11; University of Iowa, 11; Lehigh, 5; Northwestern, 12; Ohio State, 11; St. Louis, 18; Virginia, 11. But even these figures are misleading; the information indicates that most of these *students* are actually part-time or full-time *instructors* or *research workers*. A careful analysis of the information indicates that only two of these ten institutions are giving essentially a full program of graduate courses to full-time students. M.I.T. reports that the only students attending graduate classes are (a) undergraduates taking advanced work, (b) staff members from the mathematics or other departments, and (c) men employed on war research projects.

12. One school reported that both the mathematics club and the departmental colloquium had been discontinued; two others reported that the departmental colloquium had ceased to function. Twelve other institutions reported either fewer meetings of their club and colloquium or more programs devoted to applied mathematics and subjects related to war. These facts emphasize further the shortage of mathematicians, the extent to which they have turned from pure to applied mathematics, and their complete devotion to winning the war.

References

1. Frank B. Jewett, The Mobilization of Science in National Defense, Science, vol. 95, 1942, pp. 235–241.

2. This paper was prepared for the program of the Mathematical Association of America at a meeting planned for December 31, 1942 in New York City; the meeting was cancelled at the request of the government because of a shortage of transportation.

3. The author takes this opportunity to express his thanks to those who supplied the data for this report. The replies to his request for information were prompt and generous.

4. The University of California at Los Angeles is training many women for the aircraft industries, but the author does not know the exact number; see 6 below.

5. This work is beginning to take organized form. At New York University sixty college professors and high school teachers, formerly instructors in English, history, philosophy, education, and foreign languages, are taking intensive courses in mathematics and physics, sponsored by the U. S. Office of Education, to prepare them as teachers of these subjects; see War Emergency Courses in the University, Science, vol. 96, 1942, pp. 488–489. Several ESMWT courses of a similar nature have been reported also.

6. For more details of the work being done at UCLA see On the Educational Front, A.A.A.S. Bulletin, vol. 1, 1942, no. 9 (November), p. 70.

EQUIAREAL PATTERNS

M. A. SADOWSKY, Illinois Institute of Technology

1. Introduction. In a 3-page paper entitled "*Equiareal pattern of stress trajectories in plane plastic strain*" published in the June, 1941, copy of the Journal of Applied Mechanics, the present writer has reduced the problem of plasticity in two dimensions to pure differential geometry by establishing the following characteristic property of the orthogonal pattern of stress trajectories: using the trajectories as parametric curves in an orthogonal curvilinear system u, v with the first fundamental form

$$ds^2 = Edu^2 + Gdv^2, \tag{1}$$

we will have

$$EG = 1, \tag{2}$$

which expresses the equiareal property of the u, v system. To interpret (2) geometrically, let us compute the area A of a curvilinear rectangle as bounded by four parametric curves $u = u_1$, $u = u_2$ and $v = v_1$, $v = v_2$. The result

$$A = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \sqrt{EG - F^2}\, dudv = (u_2 - u_1)(v_2 - v_1),$$

shows that the pattern formed by parametric curves as selected by

$$u = 0,\ h_1,\ 2h_1, \cdots nh_1, \cdots,$$

$$v = 0,\ h_2,\ 2h_2, \cdots mh_2, \cdots,$$

consists of curvilinear rectangles all equivalent in area (h_1h_2 each). Figure 1 illustrates this by showing equiareally re-parametrized polar coordinates ($u = \sqrt{2r}$, $v = \theta$). Figure 2 shows an equiareal pattern formed by orthogonal logarithmic spirals.

A complete analytical determination of all equiareal orthogonal patterns is a problem of differential geometry which has not yet been solved. In the present paper, a particular solution involving an arbitrary function of one argument will be derived.

2. Derivation of fundamental equations. In order that (1) be a linear element of the euclidean plane, the Gaussian curvature of the uv-space has to vanish, which is expressed by the equation (known as the Gauss equation)

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\,\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\,\frac{\partial\sqrt{E}}{\partial v}\right) = 0. \tag{3}$$

We now introduce a new quantity, Ω, as defined by

$$\Omega = \frac{\partial\sqrt{E}}{\partial v} - \frac{\partial\sqrt{G}}{\partial u} \tag{4}$$

into the Gauss equation (3) in the following manner:

$$\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\left(\frac{\partial\sqrt{E}}{\partial v} - \Omega\right)\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\left(\frac{\partial\sqrt{G}}{\partial u} + \Omega\right)\right] = 0. \tag{5}$$

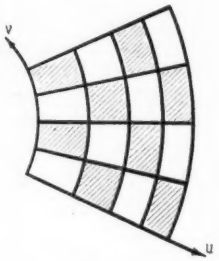


FIG. 1

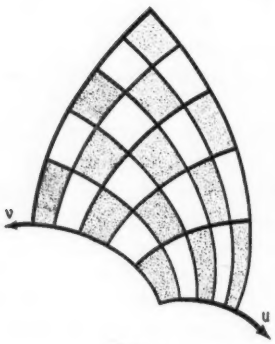


FIG. 2

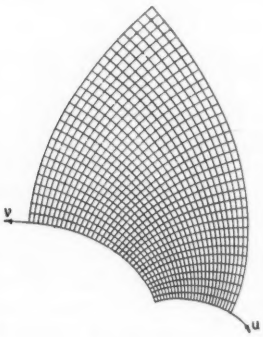


FIG. 3

Expanding (5) and using the equiareal condition (2) we may reduce the Gauss equation to

$$\frac{\partial}{\partial u}\left(\frac{\Omega}{\sqrt{E}}\right) = \frac{\partial}{\partial v}\left(\frac{\Omega}{\sqrt{G}}\right).$$

At this point we leave the general case and turn to a particular solution of the above equation as given by

$$\Omega = 0.$$

By (4), this means

$$\frac{\partial\sqrt{E}}{\partial v} = \frac{\partial\sqrt{G}}{\partial u}, \tag{6}$$

which may be integrated formally by putting

$$\sqrt{E} = \frac{\partial \phi}{\partial u}, \qquad \sqrt{G} = \frac{\partial \phi}{\partial v}, \tag{7}$$

in which $\phi = \phi(u, v)$ is an arbitrary function of u and v. The equiareal condition (2) expressed in terms of ϕ becomes

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v} = 1. \tag{8}$$

To solve (8) generally we have at our disposal a complete solution (a solution with two arbitrary constants), easy to guess:

$$\phi = au + \frac{1}{a} v + b.$$

Putting $b = f(a)$, in which $f(a)$ stands for an arbitrary function of a, and eliminating a from

$$\begin{aligned} \phi &= au + \frac{1}{a} v + f(a), \\ 0 &= u - \frac{1}{a^2} v + f'(a), \end{aligned} \tag{9}$$

we obtain ϕ in terms of u and v as expressed by $\phi = \phi(u, v)$. However, as we are not interested in $\phi(u, v)$ itself, but in its partial derivatives (7) only, we will go on as follows: assuming $a = a(u, v)$ to be the solution of the second equation in (9) for a in terms of u and v, we obtain from (7) and (9)

$$\begin{aligned} \sqrt{E} = \frac{\partial \phi(u, v)}{\partial u} &= \frac{\partial \phi(u, v, a)}{\partial u} + \frac{\partial \phi(u, v, a)}{\partial a} \cdot \frac{\partial a(u, v)}{\partial u} \\ &= a + \left(u - \frac{1}{a^2} v + f'(a) \right) \frac{\partial a}{\partial u} = a, \end{aligned}$$

which can be written in a more complete form by (2) as

$$\sqrt{E} = a(u, v), \qquad \sqrt{G} = \frac{1}{a(u, v)}. \tag{10}$$

As for $a(u, v)$, the second equation (9) may be written as

$$au - \frac{1}{a} v = F(a), \tag{11}$$

in which $F(a)$ symbolizes the arbitrary function of a. Now, the function $a(u, v)$ in (10) stands for the solution of (11) for a in terms of u and v.

To imbed the curvilinear system u, v in a Cartesian system of reference x, y we have to integrate by quadratures

$$dx + idy = \sqrt{E}\, e^{i\omega} du + i\sqrt{G}\, e^{i\omega} dv, \tag{12}$$

in which $\omega = \omega(u, v)$ stands for the local, at the point (u, v), directional angle of the parametric u-curve relative to the x-axis. The general differential geometric relation

$$d\omega = -\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} du + \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} dv,$$

can be shown to lead under (2) and (6) to

$$\omega = \log \sqrt{E}, \tag{13}$$

(an arbitrary constant has been dropped from the above because it would not be essential). This transforms (12) to

$$dx + idy = (\sqrt{E})^{1+i} du + i(\sqrt{G})^{1-i} dv, \tag{14}$$

which is ready for integration by quadratures, provided that $\sqrt{E}$ and $\sqrt{G}$ have been expressed in terms of u and v according to (10) and (11).

3. A geometric derivation of the solution. Eqs. (11), (10) and (14) define the solution analytically. A purely geometrical derivation can be made by using the following property of the solution: the trajectories intersecting the parametric u-curves at $+45°$ (bisecting the first and third quadrants formed by the parametric curves) are straight lines. The arbitrary function $F(a)$ in (11) corresponds to an arbitrary envelope of those lines. Hence, if an arbitrary one-parameter system of straight lines be given, its $\mp 45°$ trajectories will form an orthogonal equiareal system u, v of the variety as defined by eqs. (11), (10) and (14). Figure 2 belongs to this type, but Figure 1 does not, which reminds us of the fact that the solution found is but a particular one, far from being the general solution of the orthogonal equiareal pattern problem.

To establish the above geometric property by proof, let us determine the isoclinics, $\omega(u, v) = \text{constant}$. Let $\theta = \theta(u, v)$ be the angle as formed by the parametric u-curve and the isoclinic at a point (u, v). By (13), $\sqrt{E}$ is constant along an isoclinic, whence

$$\frac{\partial \sqrt{E}}{\partial u} du + \frac{\partial \sqrt{E}}{\partial v} dv = 0,$$

for a displacement from (u, v) to $(u+du, v+dv)$ along an isoclinic. The lengths of displacements in parametric directions are $\sqrt{E}du$ and $\sqrt{G}dv$. Computing θ and using (2) and (6), we have

$$\tan\theta = \frac{\sqrt{G}\,dv}{\sqrt{E}\,du} = -\frac{\sqrt{G}\dfrac{\partial\sqrt{E}}{\partial u}}{\sqrt{E}\dfrac{\partial\sqrt{E}}{\partial v}} = \frac{\dfrac{\partial(\sqrt{E}\sqrt{G})}{\partial u} - \sqrt{G}\dfrac{\partial\sqrt{E}}{\partial u}}{\sqrt{E}\dfrac{\partial\sqrt{E}}{\partial v}} = 1,$$

whence

$$\theta = 45° + k\cdot 180°. \tag{15}$$

Now, as ω is constant along an isoclinic (by the very definition of an isoclinic), $\omega+\theta$ will be constant by (15). But $\omega+\theta$ is the direction angle of the isoclinic itself relative to the x-axis. In other words: the direction of an isoclinic is invariably the same at all points of that isoclinic. This proves that the isoclinics are straight lines. Because of (15) they are the $+45°$ trajectories of the parametric u-curves, which completes the proof.

4. The "equimileage" property. Let us mention briefly a peculiar property of the solution—an "equimileage" property—which certainly is in no general relation to the equiareal property: choosing an arbitrary finite selection of parametric curves as a map of one-way streets in a curvilinear city block system (mediaeval European) to drive in the direction of increasing coordinates only as stated by the following driving rule:

$$\begin{aligned} &\text{either } du = 0 \text{ and } dv > 0, \\ &\text{or } dv = 0 \text{ and } du > 0; \end{aligned} \tag{16}$$

we have for the mileage, s, of a ride from $A(u_1, v_1)$ to $B(u_2, v_2)$

$$s = \int ds = \int \sqrt{E du^2 + G dv^2} = \int (\sqrt{E}\,du + \sqrt{G}\,dv)^*$$

$$= \int \left(\frac{\partial\phi}{\partial u}\,du + \frac{\partial\phi}{\partial v}\,dv\right) = \int d\phi = \phi_2 - \phi_1.$$

which is independent of the itinerary chosen (independent of the number and locations of the right and left turns made). We now have a geometric interpretation of the function $\phi(u, v)$: it is the "mileage function" of the map (taxi-cab fare function). Reversing all traffic against arrows, we still preserve the equimileage property with $-\phi(u, v)$ as mileage function. A partial reversal of traffic (on u-streets only, or on v-streets only) would destroy that property, making short-cuts possible since $ds = \pm(\sqrt{E}du - \sqrt{G}dv)$ is not integrable. For illustration, see Figure 2.

---

* This, I believe, is true because of (16).

5. Examples. I. Assuming $F(a)=0$ in (11), we get

$$a=\sqrt{\frac{v}{u}}, \qquad \sqrt{E}=\sqrt{\frac{v}{u}}, \qquad \sqrt{G}=\sqrt{\frac{u}{v}}.$$

An integration of (14) gives the Cartesian coördinates

$$x=\sqrt{2uv}\cos\left(\log\sqrt{\frac{v}{u}}+45°\right),$$

$$y=\sqrt{2uv}\sin\left(\log\sqrt{\frac{v}{u}}+45°\right).$$

From the above, we readily obtain the polar coordinates

$$r=\sqrt{2uv}, \qquad \theta=\log\sqrt{\frac{v}{u}}+45°,$$

whence, by (13),

$$\omega=\log\sqrt{\frac{v}{u}}=\theta-45°.$$

The parametric uv-curves are logarithmic spirals intersecting the isoclinics $\theta=$constant at $\mp 45°$ respectively. The envelope of the isoclinics, the point $r=0$, is improper. A random set of parametric curves is shown in Figure 3, while a selection forming an equiareal pattern is seen in Figure 2.

II. Assuming in (11) $F(a)=2c$ (with c a positive constant), we get

$$a=\sqrt{E}=\frac{c+\sqrt{c^2+uv}}{u}, \qquad \sqrt{G}=\frac{-c+\sqrt{c^2+uv}}{v},$$

$$\omega=\log\frac{c+\sqrt{c^2+uv}}{u}.$$

Integration of (14) gives here

$$x=\sqrt{2}\,c\cos(\omega+135°)+\sqrt{2}\sqrt{c^2+uv}\cos(\omega+45°)$$

$$y=\sqrt{2}\,c\sin(\omega+135°)+\sqrt{2}\sqrt{c^2+uv}\sin(\omega+45°)$$

from which we obtain the polar coordinates r and θ

$$r=\sqrt{4c^2+2uv}, \qquad \theta=\omega+45°+\arcsin\frac{c\sqrt{2}}{r}.$$

Using the last equation, it can be shown that the envelope of the rectilinear isoclinics is a circle of radius $c\sqrt{2}$.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON ELLIPTIC INTEGRALS

IVAN NIVEN, Purdue University

It is common knowledge that every elliptic integral is expressible in terms of integrals of three basic types.* It does not seem to have been noted, however, that every elliptic integral is expressible in terms of integrals of the so-called third type. We prove this by showing that the first and second types are dependent upon the third.

If we write

$$P = (at^2 + b)(ct^2 + d), \qquad abcd \neq 0,$$

then the three types of elliptic integrals can be written, respectively,

$$(1) \qquad I_1 = \int P^{-1/2}dt, \qquad I_2 = \int t^2P^{-1/2}dt, \qquad I_3(N) = \int (1 + Nt^2)^{-1}P^{-1/2}dt.$$

We begin with the identity

$$\frac{d}{dt}(t^3P^{-1/2}) = t^2P^{-1/2} + (d/c + b/a)P^{-1/2} - d/c(1 + ct^2/d)^{-1}P^{-1/2} - b/a(1 + at^2/b)^{-1}P^{-1/2},$$

and integrate to get

$$(2) \qquad t^3P^{-1/2} = I_2 + (d/c + b/a)I_1 - d/cI_3(c/d) - b/aI_3(a/b).$$

Similarly we integrate the identity

$$\frac{d}{dt}(tP^{-1/2}) = -P^{-1/2} + (1 + ct^2/d)^{-1}P^{-1/2} + (1 + at^2/b)^{-1}P^{-1/2}$$

to get

$$(3) \qquad I_1 = -tP^{-1/2} + I_3(c/d) + I_3(a/b).$$

Eliminating I_1 between (2) and (3) we obtain

* Cf. W. E. Byerly, Integral Calculus, 2nd edition, p. 217; or G. Mittag-Leffler, An introduction to the theory of elliptic functions, Annals of Mathematics, vol. 24 (1922–23), p. 292; or Whittaker & Watson, Modern Analysis, 4th edition, p. 515.

$$I_2 = t^3P^{-1/2} + (d/c + b/a)tP^{-1/2} - b/aI_3(c/d) - d/cI_3(a/b). \tag{4}$$

The trigonometric forms of these integrals are

$$E_1 = \int Q^{-1/2}d\phi, \qquad E_2 = \int Q^{1/2}d\phi, \qquad E_3(N) = \int (1 + N \sin^2\phi)^{-1}Q^{-1/2}d\phi$$

where

$$Q = 1 - k^2 \sin^2\phi, \qquad k^2 = bc/ad.$$

We obtain results corresponding to (3) and (4) by integrating the identities

$$\frac{d}{d\phi}(Q^{-1/2} \tan\phi) = -Q^{-1/2} + Q^{-1/2}(1 - k^2 \sin^2\phi)^{-1} + Q^{-1/2}(1 - \sin^2\phi)^{-1},$$

and

$$\frac{d}{d\phi}(k^2Q^{-1/2} \sin\phi \cos\phi) = Q^{1/2} - (1 - k^2)(1 - k^2 \sin^2\phi)^{-1}Q^{-1/2}.$$

Thus we obtain the results

$$E_1 = -Q^{-1/2} \tan\phi + E_3(-k^2) + E_3(-1),$$

and

$$E_2 = k^2Q^{-1/2} \sin\phi \cos\phi + (1 - k^2)E_3(-k^2).$$

APPROXIMATIONS TO A CENTRAL ANGLE

J. M. Bruce, Chicago, Ill.

Problem 1. To compute a central angle in terms of its arc and the height of its arc.*

Let the central angle θ of a circle of radius r intercept an arc of length a and height h. (See figure.) Then

$$h = r(1 - \cos\theta/2), \qquad r = a/\theta,$$

so that

$$\omega = h/a = (1 - \cos\theta/2)/\theta,$$

$$\omega = \theta/8 - \theta^3/384 + \theta^5/46080 + \cdots. \tag{1}$$

We take as an approximation

$$\omega = \theta/8 - \theta^3/384. \tag{2}$$

* An approximation to a central angle in terms of its chord and the chord of half its arc has been given by the author; this Monthly, vol. 49, 1942, p. 184.

To solve this cubic put $\theta = 8 \sin \phi/3$. Then (2) becomes

$$\omega = \sin \phi/3 - \tfrac{4}{3} \sin^3 \phi/3,$$

or

$$3\omega = \sin \phi.$$

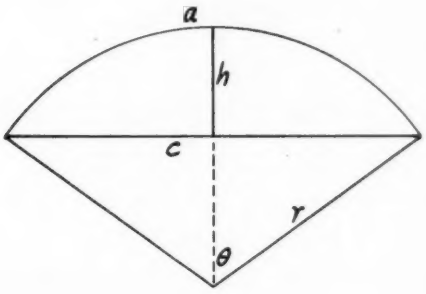
Hence

$$\theta = 8 \sin (\tfrac{1}{3} \text{ arc sin } 3\omega)$$

is an approximate solution of (1).

The error is about 6′ when $\theta = 90°$.

Problem 2. To compute a central angle in terms of its arc and the chord of its arc.



Letting c be the chord, we have

$$c = 2r \sin \theta/2,$$

which gives, on putting $\theta = 4\phi$,

$$\eta = c/a = (\sin 2\phi)/2\phi,$$

$$\eta = 1 - 2\phi^2/3 + 2\phi^4/15 + \cdots.$$

Then

$$-1 + \sqrt{1 + 3\eta} = 1 - \phi^2/2 + 3\phi^4/80 + \cdots.$$

Also

$$\cos \phi = 1 - \phi^2/2 + \phi^4/24 + \cdots.$$

Hence we take for an approximation

$$\cos \phi = -1 + \sqrt{1 + 3\eta}.$$

This gives an error in θ of about 4′ when $\theta = 90°$.

Note by the Editor. The method used by Mr. Bruce can be extended so as to give almost any degree of approximation. The method depends essentially on the selection of a function whose series expansion begins with the same terms as the given series. Thus in Problem 1, we may consider

$$\alpha \sin \beta\theta = \alpha\beta\theta - \frac{\alpha\beta^3}{6}\theta^3 + \frac{\alpha\beta^5}{120}\theta^5 + \cdots .$$

If we put

$$\alpha\beta = \frac{1}{8}, \qquad \frac{\alpha\beta^3}{6} = \frac{1}{384},$$

so that

$$\alpha = \beta = \frac{1}{2\sqrt{2}},$$

we have

$$\frac{1}{2\sqrt{2}} \sin \frac{\theta}{2\sqrt{2}} - \omega = -\frac{\theta^5}{184,320} + \cdots . \tag{3}$$

Hence we have approximately

$$\theta = 2\sqrt{2} \operatorname{arc\,sin} 2\sqrt{2}\,\omega. \tag{4}$$

To find the approximate error in θ we note that (3) can be written in the form

$$\sin \frac{\theta}{2\sqrt{2}} = 2\sqrt{2}\,(\omega - \epsilon),$$

where ϵ is about $\theta^5/184,320$. Hence

$$\begin{aligned}\theta &= 2\sqrt{2} \operatorname{arc\,sin} 2\sqrt{2}\,(\omega - \epsilon) \\ &= 2\sqrt{2} \operatorname{arc\,sin} 2\sqrt{2}\,\omega - 2\sqrt{2}\,\frac{1}{\cos \dfrac{\theta}{2\sqrt{2}}} \cdot 2\sqrt{2}\,\epsilon + \cdots ,\end{aligned}$$

so that the error in (4) is roughly

$$8\epsilon = \frac{\theta^5}{23,040}.$$

This is about 1.5′ when $\theta = 90°$.

A still better approximation can be obtained by comparing the expansions in powers of θ, of $\alpha \sin \beta\theta$ and $\sinh \gamma\omega$. For $\alpha = 1/\sqrt{3}, \beta = 1/\sqrt{23}$, $\gamma = \sqrt{23/192}$, the first three terms of the series agree, and we get the approximation

$$\theta = \sqrt{192/23} \arcsin (\sqrt{23} \sinh \omega/\sqrt{3}),$$

with an error of about $\theta^7/800{,}000$, or $6''$ when $\theta = 90°$.

In Problem 2 we can use

$$(\cos \alpha\theta)^\beta = \left(1 - \frac{\alpha^2}{2}\theta^2 + \frac{\alpha^4}{24}\theta^4 - \frac{\alpha^6}{720}\theta^6 + \cdots\right)^\beta$$

$$= 1 - \frac{\beta\alpha^2}{2}\theta^2 + \left(\frac{\beta\alpha^4}{24} + \frac{\beta(\beta-1)\alpha^4}{8}\right)\theta^4$$

$$- \left(\frac{\beta\alpha^6}{720} + \frac{\beta(\beta-1)\alpha^6}{48} + \frac{\beta(\beta-1)(\beta-2)\alpha^6}{48}\right)\theta^6 + \cdots.$$

If we put

$$\frac{\beta\alpha^2}{2} = \frac{1}{24}, \qquad \frac{\beta\alpha^4}{24} + \frac{\beta(\beta-1)\alpha^4}{8} = \frac{1}{1920},$$

which gives

$$\beta = \frac{5}{3}, \qquad \alpha = \frac{1}{\sqrt{20}},$$

we have

$$\left(\cos\frac{\theta}{\sqrt{20}}\right)^{5/3} - \eta = \frac{\theta^6}{1134000} + \cdots.$$

Thus, approximately,

$$\theta = \sqrt{20} \arccos \eta^{3/5}.$$

As in Problem 1 we find that the error in this approximation is about

$$\sqrt{20}\,\frac{1}{\dfrac{\theta}{\sin\dfrac{\theta}{\sqrt{20}}}}\,\frac{3}{5}\,\eta^{-2/5}\,\frac{\theta^6}{1134000},$$

which is roughly $\theta^5/94{,}500$ for fairly small values of $\theta/\sqrt{20}$. When $\theta = 90°$, this is about $23''$.

In general the success of this method depends mainly on one's ingenuity in concocting functions with suitable series expansions.

R.J.W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry. By Roscoe Woods. New York, The Macmillan Company, 1939. 14+294 pages. $2.25.

This book is designed as a text for colleges and technical schools; it emphasizes the conservative point of view. With the exception of two chapters, the treatment is confined to rectangular cartesian coordinates. The discussion of the problem of the distance from a line to a point is treated in the usual way, without clinching the significance of the sign, a feature of prime importance in curve tracing. Pencils of lines and of circles, including the radical axis, are discussed, but not of other conics.

The conics are introduced separately, each for itself, but a later chapter treats the general case in a comprehensive way, including poles and polars. A commendable discussion of polar coordinates makes no use of cartesian or other coordinates until the point of view is established independently. Similarly for equations in parametric form.

About fifty pages are devoted to solid analytic geometry. The topics considered include direction numbers, plane and line, and surfaces and curves.

The exercises are numerous, and graded. Answers to the odd-numbered ones are provided, and the others can be obtained from the publisher. The printing, proof-reading, and figures are all well done.

VIRGIL SNYDER

Elements of Spherical Trigonometry. By J. E. Thompson. New York, D. Van Nostrand Co., 1942. 12+144 pages. $1.65.

This book begins by summarizing, with a brief description but no proofs, everything usually studied in a Freshman Course in Plane Trigonometry. It reviews those parts of Solid Geometry which pertain to trihedral angles and spheres. The Law of Cosines for Spherical Triangles is derived from the usual trihedral angle, and all the other formulas are obtained algebraically from this one. A direct geometric proof of the Law of Sines is added as an alternative. Napier's Rules are derived as special cases of the formulas for the general triangle. All triangle solutions are arranged for logarithmic computation, and excellent model forms are given. Checking the solution is stressed in the text, but omitted entirely in the model forms.

There are numerous applications to Solid Geometry, Navigation, and Astronomy. General formulas are derived for the mensuration of a parallelepiped and the regular polyhedrons. A detailed explanation of latitude and longitude

accompanies the problems of great circle sailing. Legendre's Theorem for small triangles is developed and used to obtain the approximate area of a small triangle on the earth, but nothing is said of the accuracy of the approximation. The astronomical applications include finding the time and azimuth of sunrise or sunset, the latitude and local time from a star, and the azimuth and altitude of a star from a given point. For each of these problems, the formula which gives the desired quantity directly in terms of the known one with a minimum of computation, is derived. The right triangle is used very slightly in these applications. A good list of exercises follows each of the last four chapters. The appearance of the book could have been improved by the judicious use of bold face type. There is an excellent index and answers are given for all numerical exercises In spite of its fairly small size, it contains much more Spherical Trigonometry than is actually a prerequisite for Celestial Navigation or Astronomy.

C. E. RHODES

Introduction to the Calculus. Part I. By Samuel Beatty and J. T. Jenkins. Toronto, The University of Toronto Press, 1938. 650 pages, with index.

The aim of this text is clearly shown by two quotations from the authors' preface. "The authors hope to find a place for a type of treatment, which is meant for the student in the first instance and which contains references to the background and applications of the subject, such as are supplied ordinarily by the instructor." Later, "Naturally, this adds to the length of the treatment, without increasing the scope." The reviewer admits frankly that for his own classroom instruction, he prefers to make his own selections from background and applications, choosing them and adapting them for their usefulness and value to the particular class before him at the moment. No doubt the majority of his selections are from material covered in the present book, but a certain pedagogical value accrues in the spontaneous presentation of illustrative material to a class apart from assigned text-book sections. Without much doubt adequate presentation of the material can be found elsewhere in less than six hundred and fifty pages, but the length is due not entirely to the authors' policy of full explanation and a wealth of illustration, but also to the generous spacing of the briefest equations and expressions which adds materially to the clarity of the presentation. The authors and the University of Toronto Press should be complimented for unusually excellent mathematical typography. Misprints are very few and for the most part almost self-correcting. It is unfortunate that figure 40 does not correspond to its discussion in the text, and that orthogonality is not more apparent in figure 89. On page 107 the phrase "acceleration-component" is twice used where "velocity-component" is intended.

If the book be compared with one volume texts in introductory calculus by American authors with which the reviewer is familiar, the adequacy of its scope is apparent. The usual chapters on partial differentiation, multiple integration, and the solution of elementary differential equations are missing, but are to be supplied in the second volume of the authors' work. The reviewer observed the

absence of long lists of word problems in maxima-minima and related rates with surprise mingled with relief. On the other hand, the discussion of velocities and accelerations is carried out in great detail (perhaps even to prolixity) and followed by a treatment of moving axes in the two dimensional case. The practical possibilities of integration in terms of the elementary functions are explored to the limit. The discussion of moments is far more complete than that found in most texts. Certain topics are given completeness of treatment of a type which American authors are accustomed to leave to texts on the theory of functions of real variables. For example, the formula for the derivative of a function of a function is proved with full attention to the case where the increment of the intermediate function may have an infinite number of zeros. As a preliminary to Rolle's Theorem it is proved that a function continuous on a closed interval must take on its maximum and minimum value, and any intermediate value for a value of the independent variable belonging to the interval. The technique of operations with series—the four fundamental operations, and also differentiation, integration, and inversion of power series—is well developed. Perhaps most admirable is the workmanlike proof of the fundamental theorem of the integral calculus without dependence on the fact that both the definite integral of a function over an interval and the usual limit of the sum are measures of an area.

Partly because of the advanced nature of these topics, the reviewer regrets the lack of emphasis on definitions and on fundamental assumptions where these are used. Nowhere is there explicit formulation of the meaning of the statement that the independent variable x approaches the constant c as a limit, though "when we say that x approaches c, we mean that it does everything but arrive at c." On the other hand, the limit needed in the definition of continuity is adequately cared for, though not in the Cauchy-Weierstrass notation. Again, in Chapter III, "it is known that the ratio of length of path to length of chord approaches 1 as limiting value, as the portion of the path is shortened indefinitely." This is so fundamental an assumption that its casual introduction is a matter of regret. Almost equally a matter of pleasure is the later reduction to this very assumption of the proof that the limit of the ratio of an infinitesimal angle measured in radians to its sine is unity. In Chapter V occurs this statement, "There is a single point c common to all the closed ranges of the sequence of closed ranges

$$(a, b), \left(a, \frac{a+b}{2}\right), \left(\frac{3a+b}{4}, \frac{a+b}{2}\right), \cdots."$$

Here is a basic theorem of infinitesimal analysis repeatedly used which should either be proved or be called to the reader's attention as unproved with reasonable emphasis.

To the reviewer the order of development of the subject seems quite needlessly inverted in several respects. Why should practically all of the usual applications of differentiation (or derivation, to use the authors' phrase) be

demonstrated before the student is made acquainted with any formulas of differentiation? Why should a large number of the common applications of integration be treated three separate times, first under the head of "anti-derivation," second, as indefinite integration, and last from the point of view of the fundamental theorem? What purpose is served by intricate operations on series, even on simply convergent series, before an adequate treatment of convergence?

Assuming that the companion volume, Part II, is of similar character, the title used, "An Introduction to the Calculus," is surely modest. They will contain a very complete treatment of the calculus, preparing the student most excellently to pursue the more advanced branches of mathematics which have their foundation in calculus. In the present volume, a persual of the final chapter should convince the reader of this. The excellent proof of the fundamental theorem itself, with extended treatment of its applications, among which especial commendation should be given to the sections on moments, is accompanied by various special results of more than passing interest. Of these one is stated in closing: "Given a closed oval curve A, and let an inner curve B be described by a fixed point on a chord of constant length which moves about A and returns to its original position. The area between A and B is the area of an ellipse whose semiaxes are the segments of the chord."

E. S. HAMMOND

A Study of the Number Concept of Secondary School Mathematics. By H. F. Fehr. (Dissertation, Teachers College. Columbia University, 1940.) Upper Montclair, New Jersey, State Teachers College, 1940. 202 pages. $1.60.

The author begins by defining in a popular way the meaning of the terms Mathematics (Pure and Applied) and shows that by starting with a set of undefined relations—assumed for the sake of argument—we can build a logical structure of theorems. The two important concepts on which all processes depend are those of Number and Function. Function itself can in the ultimate be reduced to Number and the general concept of Number rests in turn on the properties we ascribe to the natural sequence one, two, three, · · · . Thus the concept of number is the basis of all analysis, without which all else must be meaningless.

To most people the word "number" is synonomous with the natural numbers 1, 2, 3, · · · . Even a fraction is not often conceived as a new number, but usually as a relation of two integers, as so many whole parts, when the whole has been divided into equal parts. Zero is the absence of all quantity, a symbol for nothingness. This limited knowledge of number suffices to give man all that is needed to carry on his everyday affairs.

For the person who studies more advanced mathematics, or the theory of its application, the arithmetic of the natural numbers is not sufficient. To explain the length of the diagonal of a rectangle, to describe such phenomena as velocity and force, etc., the natural numbers find used only in a very limited number of cases.

Today the world is using negative numbers as common knowledge. Even such symbols as $\sqrt{3}$, log 2, sin π, are appearing in some of the more popular scientific journals. While the number $\sqrt{-1}$ has not appeared as common knowledge, it can be expected that the history of the negative number can well be repeated in this case also.

If such numbers are to be used, it is essential that the proper concept of these numbers be taught so that the meaning put into them by the layman will lead to real service. This meaning can only come through a fundamental understanding of the operations and extensions of the number system.

Experimental evidence of practical need is one way of introducing negative numbers, but the method of insuring the omnipossibility of the subtraction process is a logical way. This is the concept underlying pure mathematics. If students are to have a thorough knowledge as a basis for future study of mathematics, the logical concepts must find a place in their instruction. It is the purpose of this study to show historically, the practical and logical extension of the number system of elementary secondary algebra, and to create for the teachers of elementary algebra a logical development consistent with modern interpretation.

Many teachers of mathematics have no clear idea of the number concept. The author shows by examples taken from commonly used text-books that the fault is often due to the lack of knowledge of the correct principles which define the number concept.

The following paragraph from the introduction gives clearly the method to be used:

"The logical basis for extending a number system is thru the omnipossibility of certain operations. The operations of high school algebra are addition, subtraction, multiplication, division, involution, and evolution, and these are the only operations here considered. These operations, with the notion of order are sufficient to give a clear understanding of all numbers involved in high school mathematics."

The particular purpose of this study is to create for the teachers of secondary mathematics a practical and a logical concept of the number system of high school mathematics that will enable them to impart their subject with better understanding and more lasting values.

In Chapter II is investigated the historical development of number. Each number system is traced both logically and practically through all the different civilizations. First is investigated the so-called natural numbers; then in turn are treated the fractions, negatives, rationals, real and complex numbers. In Chapter III there is similarly investigated the operations on number, the inverse of these operations and the fundamental laws of number. Particular attention is paid to the law of permanence of mathematical form.

In Chapter IV, there is given a survey of the nature of a logical development, and some recent studies on mathematical thought as applied to the number system. Some of the logical difficulties of the number concept and definition,

especially in regard to the extension of number, are examined but in no manner is a solution of such problems undertaken. The references will indicate to the reader where further study material of this nature is to be found. In Chapter V there is created a complete logical "postulate-definition-theorem-proof" development of the number system of high school algebra. From this study and the writer's own experience, there has resulted a suggested procedure for teaching the various numbers in a meaningful manner. These suggestions and criticisms of current practice are given in Chapter VI.

The above suggestions will indicate the great value of this doctor's thesis to teachers of elementary and high-school mathematics, especially to those who have had little function theory. The author gives a very complete bibliography of the subject and points out the weaknesses in many of the high-school texts.

E. B. ESCOTT

Symmetric Functions in the Theory of Integral Numbers. By Hansraj Gupta. Lucknow University Studies, No. XIV. 1940. 7+105 pages.

The text represents a short study in certain theorems in elementary theory of numbers and their generalizations. These theorems are for the most part in the nature of congruences, such as Fermat's Theorem: $a^{\phi(n)} \equiv 1 \pmod{n}$ for $(a, n)=1$ and Wilson's Theorem: $(p-1)! \equiv -1 \pmod{p}$ for p prime. The book uses a rather general method for obtaining generalizations of theorems of this type and other interesting congruences. The results are partly known and partly new. Many of the former are contained in Hardy and Wright's Introduction to the Theory of Numbers, in particular, in Chapters VII and VIII of that book.

The author introduces the function $G(n, r)$ which is defined for $0<r\leqq n$ as the rth elementary symmetric function of the n first positive integers. Thus Wilson's Theorem can be written in the form $G(p-1, p-1) \equiv -1 \pmod{p}$. Similarly, a theorem by Lagrange states that $G(p-1, r) \equiv 0 \pmod{p}$ for $p \geqq 3$ and $1 \leqq r \leqq p-2$ and Wolstenholme's Theorem says that $G(p-1, p-2) \equiv 0 \pmod{p^2}$ when $p \geqq 5$. All these theorems and more general results are obtained in the text by a detailed study of the properties of the function $G(n, r)$.

In Chapter I, the author gives a short introduction into certain aspects of the elementary theory of numbers, which need not be mentioned here in detail. He introduces the notation $\text{pot}_p n$ (p-potency of n) for the highest power of the prime p that divides n and $\text{piq}_p n$ for the highest positive exponent β (if one exists) such that $\phi(p^\beta)$ divides n. Theorems concerning the p-potency of certain binomial coefficients are derived.

Chapter II deals with the theory of primitive roots, with the order of a number a modulo m, if $(a, m)=1$, and with the *k-ic* residues modulo m. The well-known results concerning these topics are derived. It might be said at this point that some of the proofs in this chapter lack clearness, such as the proofs of Theorems 16 and 17 on pp. 26–28.

Chapter III introduces the function $G(n, r)$, mentioned above; recursion

formulas for this function are derived such as the relation $G(n, r)-G(n-1, r)=nG(n-1, r-1)$, later referred to as the "fundamental relation." The definition of $G(n, r)$ is first extended to all values of n and later to all values of r. Of importance for later applications are a number of theorems, concerning the values of the G-function for negative integral n. At the end of the Chapter, the function $S(n, r)$ (sum of the rth powers of the first n positive integers) and through it the Bernoulli numbers are expressed in terms of the G-function.

The final Chapter IV starts with an extension of the original definition of $G(n, r)$ (see above) and of $S(n, r)$ to $G(A, r)$ and $S(A, r)$, where A is a set of k non-negative integers. In this way more general sets of numbers can be considered than the set of the first n positive integers, for example, the set of the positive integers smaller than and relatively prime to n. If the corresponding functions for the latter set are denoted by $G'(n, r)$ and $S'(n, r)$, respectively, then certain congruences of these functions modulo n^2 are obtained. These congruences represent a generalization of Leudesdorf's Theorem: if $(n, 6)=1$ then $G'(n, \phi(n)-1)\equiv 0 \pmod{n^2}$. A very general recursion formula for the G-function leads through specialization to the Theorems of Lagrange, Wilson, Wolstenholme and Fermat which are thus shown to spring from the same source. In the last articles divisibility properties of $S'(n, r)$ and $S(n, r)$ are derived, also the well-known theorem of von Staudt concerning the denominators of the Bernoulli numbers. Finally Wilson's Theorem is generalized to such results as $G(p^u-1, \phi(p^u))\equiv -1 \pmod{p}$ and similar ones.

The text contains in the form of a summary a list of symbols, terms, main results, theorems. This seems almost necessary in view of the density of notations, definitions and theorems throughout the book.

The author states at the beginning that the text is written for students who have "gone through a course of numerical arithmetic," that seems to correspond approximately to a course in college algebra. While this may be theoretically true, it can hardly be expected that a student without some experience in theory of numbers and, more generally, in advanced mathematical methods should be able to follow the book without a good deal of trouble.

For anyone interested in the part of the theory of numbers, dealt with in it, Gupta's book certainly constitutes very interesting, though by no means easily digestible, reading.

FRITZ HERZOG

Fundamentals of Radio. By E. C. Jordan, P. H. Nelson, W. C. Osterbrock, F. H. Pumphrey, and L. C. Smeby. W. L. Everitt, Editor. 1 vol., Prentice-Hall, New York, 1942. 400 pages, $5.00.

The purpose of this volume is to present the basic material of radio for civil or military work. It covers the field very completely and thoroughly, and gives up-to-date information on all matters concerning long, medium and short waves. The ultra short waves or "microrays" are not included. The volume starts with a short summary of mathematics needed in radio. This is very elementary

indeed: a few notions of algebra, trigonometry, vectors and logarithms—no definition of the derivative of a function, no integrals, not even the imaginaries. With such a meager background of mathematics, it is easy to believe that most of the book will be purely descriptive, with a great many statements given without any proof. It is remarkable to see how much of the radio problems the authors are able to explain and discuss under such circumstances. They first start with a short summary of the main laws of electricity, for D.C. and A.C. circuits. A number of drawings and curves are given, to explain fundamental physical laws which should require some higher mathematics for a more precise statement.

Looking at this point of the book, and remembering some others built on the same scheme, one can hardly avoid the conclusion that it would have been much easier for the reader to learn first, and once for all, the meaning of a derivative, than to have to plot curves, draw their tangents, and discover again and again the role played by the "rate of change" of some quantities. In this review of the basic notions of electricity, it is very surprising that nothing is said about electrostatics. How can the reader understand anything on condensers, vacuum tubes, cathodic oscillographs, etc., when the only thing he is told about positive and negative electric charges is (p. 97):

> One of the most fundamental of all electrical concepts is the repelling action of charges of same polarity and the attractive forces between charges of opposite polarity.

This short sentence seems to be the only one in which electrostatic laws are summarized. It is to be hoped that the reader already has some notions of his own; otherwise he might find it hard to understand the properties of the electrons and electronic devices of all sorts.

The laws of electric currents are based on the hydraulic analogy, the volt being called "electric pressure," while a magnetic field is baptized "magnetic pressure"—and this is quite a reasonable way of introducing these notions by comparisons with the laws of liquids. There is a rather loose use of the word "flux," when (p. 61) the authors speak of the "lines of magnetic flux" meaning magnetic induction. The definition of a self-induction coefficient is given (p. 85) without any reference to the magnetic flux in the coil, but this notion appears suddenly on p. 114 for the mutual induction, where it is almost impossible to avoid. This shows how dangerous it is not to suppose a certain minimum of scientific background when discussing technics of such high standards as radio and electronics. A good working knowledge of elementary mathematics and classical physics should really be a necessary prerequisite for any student entering the field of radio; otherwise he will never be able to get any precise and accurate notion of radio itself.

The discussion and explanations of radio problems, as given in this book, are fairly complete, and supported by a great number of curves and diagrams of all sorts, which enable the authors to give a good and clear description of the most important facts and methods. Of course the whole treatment is purely

descriptive, and there are all the time a great many assumptions which the reader has to accept without discussion, as well as statements he should just memorize without any real proof of their validity. In order, however, to help the student, the authors have worked out a number of qualitative explanations which make the results quite plausible. Such methods have their advantages, but also their dangers, and some minor errors can be found, as for instance p. 124, where, speaking of the role played by the space charge in a diode, the authors write "only those electrons emitted with sufficient velocity will have enough energy to penetrate the cloud" and reach the plate. It is well known that electrons in a diode are emitted without velocity, and put into motion by the electric field inside the diode. But for this minor mistake, the chapter on electronic tubes is very good, and contains very clear descriptions of the main properties of triodes, tetrodes, and pentodes of modern designs. The discussion of the different types of amplifiers, with a clear distinction of classes *A*. *B*. *C*., and very numerous diagrams of explanation, is certainly the best part of the book. The description of wave propagation conditions is pretty good, and summarizes in twenty pages a great many valuable informations on the Heaviside layer and its practical importance. A few words on frequency modulation give a fair account of this new development of radio. The book will certainly be found very valuable for young engineers entering the field of radio, with very little scientific preparation, and it will enable them to get a fairly accurate view of the main principles and methods of radio.

L. BRILLOUIN

NEW BOOKS RECEIVED

Laboratory Geometry. By Elizabeth Roudebush. New York, Prentice-Hall, 1942. 5+192 pages. $1.12.

Basic Mathematics. By W. W. Hart. Boston, D. C. Heath and Co., 1942. 6+456 pages. $1.52.

A Review of Arithmetic. By Z.L.Smith. Chicago, Institute of Military Studies, University of Chicago, 1942. 37 pages. $0.25.

Plane Trigonometry. By E. R. Heineman. New York and London, McGraw-Hill Book Co., Inc., 1942. 12+167 pages. $2.00.

Spherical Trigonometry. By R. W. Brink. New York, D. Appleton-Century Co., Inc., 1942. 8+62 pages. $0.75.

The Non-Singular Cubic Surface. A new method of investigating with special reference to questions of reality. By B. Segre. Oxford, University Press and Humphrey Milford, 1942. 11+180 pages. $4.25.

Descriptive Geometry for Engineering and Architectural Draftsmen. By F. A. P. Fischer and J. A. Clear. Milwaukee, The Bruce Publishing Co., 1942. 7+244 pages. $3.00.

Plane and Spherical Trigonometry. By F. A. Rickey and J. P. Cole. New York, The Dryden Press, 1942. 10+209 pages. $2.25.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

SOME APPLICATIONS OF THE SLIDE RULE

A number of the club reports this year indicate an interest in the use of the slide rule. Students who have mastered the technique of reading the scales, and who know how to multiply, divide, and extract square roots, are ready to learn more tricks. The tricks shown below are but a few of the many which might serve to illustrate the power of the slide rule as a device for speeding computation.

1. Vector resultants. An important problem in physics is to determine the magnitude and direction of the resultant of two mutually perpendicular vectors. This is equivalent to solving a right triangle in which a and b are given lengths and $\angle C = 90°$. Let us assume that b is the longer of the two given sides, so that $a < b$. Then to find A and c, we use the formulas

$$\frac{\tan A}{1} = \frac{a}{b}, \qquad 1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2. \tag{1}$$

Setting a on the **C** scale opposite b on **D**, the angle A is read on the **T** scale opposite the right hand index of the **D** scale. (On some rules this is found under a special hairline on the back side.) Without changing the setting, we read $(b/a)^2$ on the **A** scale opposite the left hand index on **B**. Adding 1 to this number we obtain $(c/a)^2$. Then moving the slide to the right until the index on the **B** scale is opposite $(c/a)^2$ on the **A** scale, the hypotenuse c can be read on the **D** scale opposite the known value a on the **C** scale. Thus the magnitude and direction of the resultant vector are obtained from two settings of the rule.

Example: Given $a = 105$, $b = 208$. To find A and c. Set 105 on **C** opposite 208 on **D**. Read $\angle A = 36°47'$ on **T** opposite the right index on **D**. Also read $(b/a)^2 = 3.92$ on the **A** scale opposite the left index on **B**. Add 1. Then, setting the left index on **B** opposite 4.92 on **A**, read $c = 233$ on **D** opposite 105 on **C**.

2. Graphs of ellipses and hyperbolas. The coordinates of several points on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ may be computed easily by the slide rule, if the equation is written in the form

$$\frac{y^2}{a^2 - x^2} = \frac{b^2}{a^2}. \qquad \begin{matrix} \textbf{A scale} \\ \textbf{B scale} \end{matrix} \tag{2}$$

The slide is set either by placing a^2 on the **B** scale opposite b^2 on the **A** scale, or

by placing a on the **C** scale opposite b on the **D** scale. For various simple values of x, the values of a^2-x^2 can be computed mentally. (For small integral values of x the values of x^2 are known and for half-integers we note that $(n+\frac{1}{2})^2 = n(n+1)+\frac{1}{4}$. Other squares can be read off the slide rule itself, using the **D** and **A** scales). Then with the hairline over a^2-x^2 on the **B** scale we read the corresponding value of y on the **D** scale.

For the hyperbola $(x^2/a^2)-(y^2/b^2)=1$, we use the same set-up except that x^2-a^2 is used in place of a^2-x^2.

3. The parabola in polar coordinates. To compute the polar coordinates of points on the parabola

$$r = 2a/(1 + \cos\theta),$$

it is best to use a trigonometric identity and write

$$r = a + a\tan^2(\theta/2). \tag{3}$$

With the appropriate index of the **T** scale opposite a on the **A** scale we read $a\tan^2(\theta/2)$ on **A** opposite $\theta/2$ on **T**. Adding a, we obtain r.

4. Accurate determination of angles. It is a fact that an acute angle θ can be determined more accurately from $\tan\theta$ or $\cot\theta$ than from $\sin\theta$ or $\cos\theta$, and more accurately from $\log\tan\theta$ or $\log\cot\theta$ than from $\log\sin\theta$ or $\log\cos\theta$. It is also true that angles are determined more accurately from the **T** scale on a slide rule than from the **S** scale. Thus, given $\cos\theta=12/13$, we may read on the **S** scale that $90°-\theta=67°30'$ and hence that $\theta=22°30'$, but we are uncertain of the answer within $10'$. On the other hand, if we write $\tan^2\theta=(13/12)^2-1 = 25/144$, and set the right index of the **T** scale opposite 14.4 on the **A** scale, we find $\theta=22°37'$ on **T** opposite 2.5 on **A**, and the angle θ is accurate to the nearest minute.

As another example, let $\sin\theta=\sqrt{5/6}$. We have $\sin^2\theta=5/6$, $\cos^2\theta=1/6$, $\tan^2\theta=5$, $\tan^2(90°-\theta)=1/5$. Setting the right hand index on **T** opposite 5 on **A**, we read $90°-\theta=24°6'$ on **T** opposite 1 on **A**. Hence $\theta=65°54'$, accurate to the nearest minute. This is a much more exact result than could have been read from the **S** scale.

If the given angle θ is near 45° it is a good idea to determine θ from $\cos 2\theta$. For example, given $\sin\theta=21/29$, then $\cos 2\theta=1-2\sin^2\theta=-41/841$, and we have $\sin(2\theta-90°)=41/841$. Setting the right index of the **S** scale opposite 84.1 on the appropriate scale (**A** or **D** according to the type of slide rule) we read $2\theta-90°=2°48'$ on the **S** scale opposite 4.1, correct to within $20''$. Hence $\theta=46°24'$ correct to within $10''$.

5. Solution of a cubic equation. An irreducible cubic equation in which the quadratic term is absent may be written in the form

$$ax^3 - bx + 1 = 0, \tag{4}$$

by dividing by the constant term. The roots are the abscissas of the intersections of the two curves

$$y = ax^2 + (1/x), \qquad y = b. \tag{5}$$

A setting of the rule is made with the index of the **B** scale opposite a on the **A** scale. Then when the hairline covers x on the **C** scale it will cover ax^2 on the **A** scale and $(1/x)$ on the **CI** scale. Starting with an approximate value of x on **C**, the hairline is moved until the sum y of the readings on **A** and **CI** has the given value b. Then x is read on the **C** scale.

Example: To find the roots of the equation $3x^3-10x+6=0$, we first note that there are three roots, lying respectively between 1 and 2, between 0 and 1, and between -3 and -2. We then write, as above in (5),

$$y = 0.5x^2 + (1/x), \qquad y = 1.667. \tag{5a}$$

We set the index on **B** opposite 0.5 on **A**, and take as first approximations for the roots the values 1.25, 0.75 and -2.00, respectively. With successive approximations to each root we finally obtain in each case $y=1.667$ as the sum of the readings on **CI** and **A**.

Scale		First root				Second root			Third root		
C	x	1.25	1.40	1.36	**1.368**	0.75	0.70	**0.705**	-2.00	-2.10	$\mathbf{-2.073}$
CI	$(1/x)$	.80	.71	.735	.731	1.33	1.429	1.418	-0.50	-0.48	-0.483
A	$0.5x^2$	.78	.98	.925	.936	.28	.245	.249	2.00	2.20	2.150
Sum $=y$		1.58	1.69	1.660	1.667	1.61	1.674	1.667	1.50	1.72	1.667

Thus the roots are 1.368, 0.705, and -2.073. More accurate determinations by algebraic methods are 1.367953, 0.705219, -2.073171. As a check we use the fact that the sum of the three roots is 0.

CLUB REPORTS 1941–42

Pi Mu Epsilon, Hunter College

The topic for the year was the *Theory of Matrices*. The papers were: *Vector analysis* by Gertrude Mandel and Charlotte Brown, *Homogeneous linear equations* by Ilse Novak, *Orthogonal complements* by Lucy La Sala, *Non-homogeneous linear equations* by Edris Adams and Helen Trief, *Bilinear forms* by Mae Reiner, *The characteristic equations* by Pearl Schutz, *Determinant of a matrix* by Evelyn Levine and Barbara Samson, *Postulate systems* by Brenda Lansdown, *Matrices with polynomial elements* by Elaine Frankel, *Tschirnhaus transformations* by Blossom Klein and Julia Rubin, *Resultants and discriminants* by Harriet Eisenberg. The officers were: President, Louise Miller; Recording Secretary, Claire Schwartz; Treasurer, Hortense Schindler; Corresponding Secretary, Phyllis Monderer; Faculty Adviser, Professor C. C. MacDuffee.

Mathematics Club, Hunter College

The Mathematics Club held seventeen bi-monthly meetings this year. Faculty and visiting speakers were: Dr. Rosario Candela who spoke on *Cryptography*, Professor Marston Morse on *Equilibrium points*, Professor Jewell Bushey on *Finite geometry*, Mr. Edward Molina of the Bell Telephone Laboratories on *Applications of probability to telephone engineering*, Dr. Raymond Lorch

of Columbia University on *Rings*. Students speakers included Miss Phyllis Monderer, *Proof of the law of gravitation*, and Miss Julia Rubin, *Statistics in Psychology*. The officers: were President, Gertrude Mandel; Vice President, Rosa Weill; Secretary, Evelyn Levine; Treasurer, Julia Rubin; Publicity Manager, Dora Feldman; Assistant Publicity Manager, Harriet Eisenberg; Faculty Adviser, Professor Mina Rees.

Mathematics Club, University of Buffalo

The program for 1941–42 included the following topics: *Mathematical oddities* by Joan Searles and Wallace Barnes, *Curve fitting by the method of least squares*, the paper which won the Sherk Prize for the year 1940–41, by Ruth Euller, *The mathematics of bridge* by Robert Rittenhouse, *Codes and ciphers* by Marvin Messler, *A ladder problem* by Lois Obenauer, *Proving axioms of geometry by algebraic methods* by Jack Castle, *Aesthetic measure of polygons* by Annabel Miller, *An evening at Monte Carlo* by Wallace Barnes, *Airplane designs* by Edward Wallenhorst. Each year the club invites high school students who are interested in mathematics to attend one of the meetings at which a suitable topic is presented. Various informal games and mathematical recreations are planned after the formal meeting. The club officers would welcome an exchange of ideas with other clubs for the planning of the recreation period. Officers for 1941–42 were: President, Joseph Ullman; Vice President, Annabel Miller; Secretary-Treasurer, Ruth Brendel; Faculty Adviser, Professor Harriet Montague.

Pi Mu Epsilon, University of Alabama

The following programs were presented: *Polynomials* by Dr. Paul Hummel, *Klyston, a new ultra high frequency generator*, by Dr. F. H. Mitchell, *Unique factorization* by Dr. H. S. Thurston, *Polynomials* by Dr. R. W. Cowan, *Phases of descriptive geometry* by Dr. W. G. Warnock. The officers for 1941–42 were: Director, Ria Jane Clinkscales; Vice-Director, Louise McClanahan; Secretary, William F. Adams; Treasurer, B. G. Clark; Librarian, Dr. W. P. Ott.

Mathematics Club, Boston University

Professor J. L. Coolidge of Harvard gave a most successful talk on *How the Greeks solved quadratic equations*, and Professor R. M. Frye of the Physics Department presented a paper on *Ballistics*. There was also a joint meeting at Harvard with the Greater Boston Association of College Mathematics Clubs, at which the principal speaker was Professor E. B. Mode, who spoke on *Finite differences*. The officers for 1942–43 are as follows: President, Betty Barry, Vice-President, John Haynes; Secretary, Mary Siteman; Treasurer, Michael Argeros; Executive Committee, Daniel Feer, Earl Maby, Minnie Young; Faculty Adviser, Professor Robert Bruce.

Mathematics Club, St. John's University

The club at its first meeting of the year voted to confine its activities as far as possible to matters concerned with national defense. Accordingly lectures on navigation, surveying, exterior and interior ballistics, bombing, and the slide rule were given by various members of the faculty. One meeting was devoted to an *Information Please* program in which the club members attempted to answer mathematical questions that had been sent in by vaious students and faculty members. Officers were: President, Howard Bryant; Vice-President, Gerard Saur; Secretary-Treasurer, Gerard O'Donnell; Publicity Director, Richard Fantin; Faculty Adviser, Professor J. J. McCarthy.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 551. *Proposed by J. H. Butchart, Grinnell College*

Show that the arcs common to a circular cylinder and a right helicoid whose axis lies on the cylinder are pieces of a circular helix.

E 552. *Proposed by V. Thébault, San Sebastián, Spain*

In certain scales of notation, a number of the form $aabb$ can be the square of a number of the form cc, where c is a multiple of b. Show that (1) a and b are relatively prime; (2) b and $c^2/b-a$ are perfect squares.

E 553. *Proposed by N. A. Court, University of Oklahoma*

If two of the four circles of intersection of two spheres with two planes are cospherical, prove that the remaining two circles are likewise cospherical.

E 554. *Proposed by J. L. Woodbridge, Philadelphia*

Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

E 555. *Proposed by Howard Eves, Syracuse University*

Consider a rectangular parallelepiped of dimensions $a \times b \times c$, made up of abc unit cubes. Imagine the edges of all these unit cubes replaced by material wires, common edges sharing the same wire. Prove that the exterior surface of the resulting wire network is of genus $p=2abc+bc+ca+ab$ (*i.e.*, that it is topologically equivalent to a sphere with p handles).

SOLUTIONS

Fourth Powers of Odd Numbers

E 512 [1942, 195]. *Proposed by V. Thébault, San Sebastián, Spain*

Find the first n odd numbers whose sum divides the sum of their fourth powers.

Solution by F. A. Butter, Jr., University of Southern California

A trivial solution is given by $n=1$, so we suppose that $n>1$. Let

$$T_1(n) = 1 + 3 + \cdots + (2n - 1) = n^2,$$
$$T_4(n) = 1^4 + 3^4 + \cdots + (2n - 1)^4,$$

and

$$S(N) = 1^4 + 2^4 + \cdots + N^4 = N(N + 1)(2N + 1)(3N^2 + 3N - 1)/30.$$

Evidently

$$T_4(n) = S(2n) - 2^4S(n) = n(4n^2 - 1)(12n^2 - 7)/15.$$

We must therefore find the least integer $n>1$ such that

$$n(4n^2 - 1)(12n^2 - 7) = 15kn^2,$$

where k is an integer. This can be written

$$n(15k - 48n^3 + 40n) = 7,$$

whence $n=7$ necessarily. In fact

$$T_4(7)/T_1(7) = (1^4 + 3^4 + \cdots + 13^4)/(1 + 3 + \cdots + 13)$$
$$= 52871/49 = 1079.$$

Also solved by Adolph Barjansky, D. H. Browne, W. E. Buker, R. E. Crane, Howard Eves, J. F. Heyda, C. A. Murray, E. P. Starke, and the proposer.

Editorial Note. By the Euler-Maclaurin Summation Formula (Boole, *Finite Differences*, p. 90), we have

$$\Delta^{-1}(2x)^4 = \int (2x)^4dx - \tfrac{1}{2}(2x)^4 + \tfrac{1}{12}D(2x)^4 - \tfrac{1}{720}D^3(2x)^4$$
$$= \tfrac{16}{5}x^5 - 8x^4 + \tfrac{10}{3}x^3 - \tfrac{8}{15}x + C$$
$$= \tfrac{8}{15}x(x - 1)(2x - 1)(3x^2 - 3x - 1) + C$$

$=f(x)$, say. Therefore

$$1^4 + 3^4 + \cdots + (2n - 1)^4 = f(n + \tfrac{1}{2}) - f(\tfrac{1}{2}) = f(n + \tfrac{1}{2}) - C$$
$$= \tfrac{16}{15}n(n^2 - \tfrac{1}{4})(3n^2 - \tfrac{7}{4})$$
$$= \tfrac{1}{15}n(4n^2 - 1)(12n^2 - 7),$$

as before.

Equal Intercepts between Three Planes

E 513 [1942, 195]. *Proposed by N. A. Court, University of Oklahoma*

A line revolves about a fixed point in such a manner that the segment intercepted on it by two intersecting planes has its mid-point in a third given plane. Show that the locus of the variable line is a cone of the second degree.

Solution by Howard Eves, Syracuse University

The cone generated will be of the second degree if a general plane through the vertex cuts the cone in two, and only two, generators. Let P be the vertex and ω

a plane through P. Let ω cut the three given planes in lines a, b, c, line c lying in the third plane. Then, from the conditions of the problem, a and b (in general) intersect. Thus the desired result follows from a known theorem which tells us that in ω there are just two lines through P such that c bisects the segment cut off by a and b. This may be proved as in the *Educational Times, Reprints*, vol. 6, 1904, p. 103, Question 15456, or else as follows.

Let P have coordinates (x_1, y_1) referred to the intersecting lines a and b as (oblique) cartesian axes. Then the equation of any line through P has the form

$$y - y_1 = k(x - x_1).$$

We readily find the points where this meets the axes to be $(x_1 - y_1/k, 0)$ and $(0, y_1 - kx_1)$. The point (x, y) midway between these is given by

$$x = \tfrac{1}{2}(x_1 - y_1/k), \qquad y = \tfrac{1}{2}(y_1 - kx_1).$$

Eliminating k we obtain, as the locus of the midpoint, the conic

$$2xy - xy_1 - yx_1 = 0.$$

Any third line c can cut this conic in at most two points.

Also solved by Adolph Barjansky, L. M. Kelly, E. P. Starke, P. D. Thomas, and the proposer.

Editorial Note. The use of oblique axes suggests the following direct solution. Let the three planes be $x=f$, $y=g$, $z=h$. The line $x/l=y/m=z/n$ meets the first where $z=nf/l$, and the second where $z=ng/m$. The conditions of the problem require the mean of these two values of z to be h, whence

$$\frac{f}{l} + \frac{g}{m} = \frac{2h}{n}.$$

Thus the desired locus is the cone $f/x+g/y=2h/z$, or

$$fyz + gzx - 2hxy = 0.$$

An Exponential Series

E 514 [1942, 195]. *Proposed by J. A. Bullard, University of Vermont*

Find the sum $\sum_{k=0}^{n-1} e^{kx} \sin(y+kz)$.

Solution by B. H. Bissinger, Cornell University

Let $t=x+iz$. Then, assuming x, y, and z to be real, we see that the desired sum is the imaginary part of

$$\sum_{k=0}^{n-1} e^{kt+iy} = \frac{e^{nt}-1}{e^{t}-1}\, e^{iy} = \frac{(e^{nx+i(y+nz)} - e^{iy})(e^{x-iz}-1)}{(e^{x+iz}-1)(e^{x-iz}-1)},$$

provided $e^t \neq 1$. Thus

$$\sum_{k=0}^{n-1} e^{kx} \sin(y+kz) = \frac{e^{(n+1)x} \sin\{y+(n-1)z\} - e^{nx}\sin(y+nz) - e^{x}\sin(y-z) + \sin y}{e^{2x} - 2e^{x}\cos z + 1},$$

unless $x=0$ and z is a multiple of 2π, in which case the sum is simply $n \sin y$.

Also solved by F. A. Butter, Jr., E. P. Starke, and the proposer.

Commensurable Triangles

E 515 [1942, 195]. *Proposed by H. T. R. Aude, Colgate University.*

Find all the triangles with integral sides which have one side equal to 16 units and the cosine of an adjacent angle equal to $-\frac{1}{4}$.

Solution by E. P. Starke, Rutgers University

Let the other sides be a and b, with a opposite the stated angle. The law of cosines gives $a^2=b^2+256+8b$, or

$$(a+b+4)(a-b-4)=240.$$

If the factors of the left member are equated to the possible pairs of even integral factors of 240, the greater factor being equated to $a+b+4$, the following systems are obtained:

$$a+b+4 = 120,\ 60,\ 40,\ 30,\ 24,\ 20;$$
$$a-b-4 = \quad 2,\ \ 4,\ \ 6,\ \ 8,\ 10,\ 12.$$

These yield the pairs of values

$$(61, 55),\quad (32, 24),\quad (23, 13),\quad (19, 7),\quad (17, 3),\quad (16, 0)$$

for (a, b). There are thus exactly five non-trivial triangles of the desired kind.

Also solved by Adolph Barjansky, F. A. Butter, Jr., Howard Eves, L. M. Kelly, P. D. Thomas, and the proposer.

A Balanced Incomplete Block Design

E 516 [1942, 257]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

During one week, Mr. X invited just four of his seven friends for dinner each night. The invitations were arranged so that any given pair of guests dined together on just two occasions, and for any two given nights there were two guests who were present both times. Show how this was managed. In how many different ways can it be done?

Solution by W. B. Campbell, Philadelphia

Among the 7 guests there are $\binom{7}{2}=21$ guest-pairs, to be put in one-to-one correspondence with the $\binom{7}{2}$ night-pairs, so one of the stated requirements is redundant. If a given quadruple of guests, such as $ABCD$, appears on one night, each of the 6 pairs contained in it must also appear on just one of the other 6 nights. Any pair from the remaining guests must appear on two nights on which the two pairs from the given quadruple do not overlap, as otherwise there would be more than one repeated guest-pair for that night-pair; *e.g.*, if EF is associated with AB, it is also associated with CD. Since there are 3 available pairs in EFG, and 3 pairs of distinct pairs in $ABCD$, the association can be affected in $3!=6$ ways. Thus each of the $\binom{7}{4}=35$ available quadruples generates 6 different ar-

rangements of 7 compatible quadruples; but as each arrangement could be derived from any one of its 7 constituent quadruples, the actual number of arrangements is $35 \cdot 6/7 = 30$. Considering permutations of the quadruples in a given arrangement over the 7 nights, we find the total number of admissible lists to be

$$30 \cdot 7! = 151200$$

(still ignoring the order of seating at the table). A typical arrangement is *ABCD ABEF, CDEF, ACEG, BDEG, ADFG, BCFG.*

Also solved by E. P. Starke.

Editorial Note. This problem was taken from R. D. Carmichael, *Groups of Finite Order*, Boston, 1937, p. 25, Ex. 15, 16. A cyclic solution (in terms of the residues modulo 7) is given by

$$0124,\ 1235,\ 2346,\ 3450,\ 4561,\ 5602,\ 6013$$

and is represented graphically by the white squares of the second "anallagmatic tessellation" drawn on p. 109 of Rouse Ball, *Mathematical Recreations and Essays* London, 1939. The guests whom Mr. X did *not* invite, each night, form the famous triple system of the finite geometry $PG(2, 2)$, whose collineation group is of order 168 (Carmichael, *op. cit.*, p. 22). Therefore the number of such arrangements is $(7!)^2/168 = 151200$, as above.

Unequal Tetrahedra

E 517 [1942, 257]. *Proposed by V. Thébault, San Sebastián, Spain*

If two tetrahedra have equal areas for corresponding faces, do they necessarily have the same volume?

Solution by E. P. Starke, Rutgers University

A negative answer is indicated by specific cases. The tetrahedra with vertices

$$(0, 0, 0),\quad (2, 0, 0),\quad (0, 2, 0),\quad (\sqrt{3}, -\sqrt{3}, 1);$$
$$(0, 0, 0),\quad (2, 0, 0),\quad (0, 2, 0),\quad (1, 1, \sqrt{3})$$

have equal areas for corresponding faces (viz. $2, 2, 2, \sqrt{6}$), but the volumes are $2/3$ and $2/\sqrt{3}$. (Note that one pair of corresponding faces are congruent.) There is some interest in the tetrahedron

$$(0, 0, 0),\quad (6, 8, 0),\quad (12, 0, 0),\quad (6, -1, 3\sqrt{7}),$$

whose four faces are congruent isosceles triangles of area 48 each, and whose volume is $48\sqrt{7} = 127.0$. A regular tetrahedron of edge $8 \cdot 3^{1/4}$ has the same area for each face, but its volume is $2^{15/2}\, 3^{-1/4} = 137.5$.

Also solved by Howard Eves and the proposer.

Three Gaussian Integers

E 518 [1942, 257]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Find three Gaussian integers x, y, z which satisfy the equation

$$x^p + y^p = z^p$$

for every prime p greater than 3.

Solution by the Proposer

$$x = 1 + i\sqrt{3}, \qquad y = 1 - i\sqrt{3}, \qquad z = 2.$$

Since $x/2$ and $y/2$ are sixth roots of unity, we merely have to verify $x^n+y^n=z^n$ for $n=\pm 1$, and we can deduce the same for $n=6k\pm 1$ where k is any rational integer.

Ballistics

E 519 [1942, 257]. *Proposed by Paul Brock, Brooklyn College*

If a projectile is aimed at a given point from a given origin, find the direction to which the two possible paths are equally inclined initially.

Solution by E. A. Nordhaus, University of Wisconsin in Milwaukee

If the initial slopes of the two paths from the origin to the point (x, y) are $m_1=\tan \alpha_1$ and $m_2=\tan \alpha_2$, then m_1 and m_2 are the roots of the equation (for m)

$$x^2m^2 - kxm + x^2 + ky = 0,$$

where $k=2v_0^2/g$. (See W. D. MacMillan, *Statics and the Dynamics of a Particle*, New York, 1927, p. 254.) On using the expressions for the sum and product of the roots, we find

$$\tan (\alpha_1 + \alpha_2) = - x/y = \cot (- \theta),$$

where $\tan \theta=y/x$. Then $\alpha_1+\alpha_2=\theta+90°$, and the required direction-angle is

$$\tfrac{1}{2}(\alpha_1 + \alpha_2) = \tfrac{1}{2}\theta + 45°.$$

Also solved by Howard Eves, L. M. Kelly, G. A. Yanosik, and the proposer.

Similar Figures

E 521 [1942, 335]. *Proposed by J. R. Musselman, Western Reserve University*

(a) On the sides BC and CA of a triangle ABC, construct externally any two directly similar triangles, CBA_1 and ACB_1. Show that the midpoints of the three segments BC, A_1B_1, CA form a triangle directly similar to the two given triangles.

(b) On BC externally, and on CA internally, construct any two directly similar triangles CBA_1 and CAB_1. Show that the midpoints of AB and A_1B_1 form with C a triangle directly similar to the two given triangles.

Solution by Howard Eves, Syracuse University

The two theorems are very special cases of the fundamental theorem concerning two directly similar figures: If the lines joining corresponding points of two directly similar figures be divided proportionally, the locus of the points of division will be a figure directly similar to the given figures.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4065. *Proposed by P. Erdös, Princeton, N. J.*

(1) Let n given points have the property that the straight line joining any two of them passes through a third point of the set. Show that the n points lie on a straight line.

(2) Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines.

4066. *Proposed by Richard Bellman, Brooklyn College*

Prove that

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \frac{dx}{\Gamma(x)} + e\int_1^2 \frac{dx}{\Gamma(x)} - \frac{e}{2!}\int_2^3 \frac{dx}{\Gamma(x)} + \frac{e}{3!}\int_3^4 \frac{dx}{\Gamma(x)} - \cdots .$$

4067. *Proposed by H. S. Wall, Northwestern University*

Prove that, if $0<x<1$,

$$\prod_{n=1}^{\infty}(1-x^{2n-1}) = 1 \Big/ \left[1 + \sum_{k=1}^{\infty} x^{k(k+1)/2}/(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)\right].$$

4068. *Proposed by James Singer, Brooklyn College*

If a and n are integers, each greater than unity, and r is any positive integer, and if $\phi(x)$ is the Euler function, show that $\phi(1+a^r+a^{2r}+\cdots+a^{nr})$ is a multiple of $2r(n+1)$.

4069. *Proposed by V. Thébault, San Sebastian, Spain*

Let I_a, I_b, I_c, I_d denote the centers of the spheres escribed in the truncated trihedral angles for the corresponding vertices of the tetrahedron $ABCD$, and A_1, B_1, C_1, D_1 the points where the straight lines AI_a, *etc.* meet the faces BCD, *etc.* Show that

(1) $$V'/V = -ABCD/(S-A)(S-B)(S-C)(S-D),$$

(2) $$V_1/3V = -ABCD/(B+C+D)(C+D+A)(D+A+B)(A+B+C),$$

where V, V', V_1 denote the volumes of the tetrahedrons $ABCD$, $I_aI_bI_cI_d$,

$A_1B_1C_1D_1$; A, *etc.* denote the areas of the faces BCD, *etc.* and $2S=A+B+C+D$.

Note. The formula (2) is due to Genty, *Nouvelles Annales de Mathématiques*, 1880, p. 528 and 1881, p. 341.

SOLUTIONS

Cyclic Numbers

4012 [1941, 639]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of n digits $N_0=a_1a_2\cdots a_n$, $(a_1\neq 0)$, such that if we transpose the first k digits from left to right, $(k=1, 2, 3, \cdots, n-1)$, the $n-1$ numbers thus obtained $N_1=a_2a_3\cdots a_na_1$, $N_2=a_3a_4\cdots a_na_1a_2$, $\cdots$, $N_{n-1}=a_na_1a_2\cdots a_{n-1}$ are each multiples of N_0.

Solution by E. P. Starke, Rutgers University

If we exclude the trivial case where the digits of N_0 are all alike and $N_k=N_0$ for every k, we may show that N_0 is 142857 (the repetend of the decimal expansion of 1/7) or is a number made up of two or more repetitions of these digits.

Notice first that, for every j,

$$a_1 \leqq a_j, \tag{1}$$

for otherwise N_{j-1}, being less than N_0, is not a multiple. If a_1 is not the smallest digit of N_0, let $j+1$ be the least subscript for which $a_{j+1}=a_1$. Then N_j is neccessarily equal to N_0, and $a_{j+2}=a_2$, $a_{j+3}=a_3$, $\cdots$: N_0 consists of the repetend $N_0'=a_1a_2\cdots a_j$ written in succession two or more times; and, furthermore, N_0' has the stated property as well as N_0.

Suppose $a_1=1$. Then, because 1 is the final digit of N_1, N_1/N_0 is not an even integer and not 5. Let $N_1/N_0=3$, $(a_1=1)$. Then $a_2a_3\cdots a_n1/1a_2\cdots a_{n-1}a_n=3$ implies $a_n=7$; $a_2a_3\cdots a_{n-1}71/1a_2\cdots a_{n-1}7=3$ implies $a_{n-1}=5$; similarly $a_{n-2}=8$, $a_{n-3}=2$, $a_{n-4}=4$. If $n=6$, so that $a_{n-4}=a_2=4$ is the leading digit of N_1, the process is complete, giving $N_0=142857$. If $n>6$, the above sequence is repeated: $a_{n-5}=1$, $a_{n-6}=7$, $a_{n-7}=5$, $\cdots$, giving N_0 made up of two or more repetitions of 142857. There exists no value of a_2 (with $a_1=1$) for which $a_2\cdots/1a_2\cdots=N_1/N_0$ can equal 7 or 9.

For $a_1=2$ and $N_1/N_0=2$, $a_2\cdots a_n2/2a_2\cdots a_n=2$ implies $a_n=6$ (since by (1) $a_n\neq 1$), but there exists no a_{n-1} such that $a_2\cdots a_{n-1}62/2a_2\cdots a_{n-1}6=2$. For $a_1=2$, $N_1/N_0=3$, $a_2\cdots a_n2/2a_2\cdots a_n=3$ implies $a_n=4$ and then $a_{n-1}=1$, contrary to (1). Similar elementary considerations eliminate all other values for a_1 and N_1/N_0, and thus establish the initial statement.

Editorial Note. A generalized form of the problem is as follows:

Find the numbers $N=d_1d_2\cdots d_q$, $d_1>0$, in a scale of radix s, which do not consist of consecutive sets of the same digits in the same order, and such that

$$N_k = d_{k+1}d_{k+2}\cdots d_qd_1\cdots d_k = r_kN, \qquad k = 1, 2, \cdots, q-1, r_q = 1.$$

From the given conditions it is obvious that $q\geqq 2$ and $r_k<s$; and it can be shown by Starke's reasoning that $d_1<d_k$, $k\neq 1$, and that $r_i\neq r_j$, $i\neq j=1, 2, \cdots, q$. Also we have

$$(1)\qquad N = P_k s^{q-k} + Q_k, \qquad N_k = Q_k s^k + P_k, \qquad P_q = N, \qquad Q_q = 0,$$
$$P_k = d_1 d_2 \cdots d_k, \qquad Q_k = d_{k+1} d_{k+2} \cdots d_q.$$

From these relations we have

$$(2)\qquad N(s^k - r_k) = P_k(s^q - 1), \qquad N(r_k s^{q-k} - 1) = Q_k(s^q - 1).$$

Let u be the g.c.d. of N and $s^q - 1$ so that $N = nu$, $s^q - 1 = pu$, where n and p are relatively prime. We then have from (2)

$$(3)\qquad n(s^k - r_k) = P_k p, \qquad n(r_k s^{q-k} - 1) = Q_k p, \qquad k = 1, 2, \cdots, q,$$

and from the first equations of (3) considered in turn we must have $n \mid d_1$, $n \mid d_2, \cdots, n \mid d_q$. After removing this common divisor n of the digits, if $n > 1$, we have another number $\overline{N}$ which obviously satisfies all of the requirements; we shall suppose that this has been done and in order to avoid new notations we shall write

$$(4)\qquad s^k - r_k = P_k p, \qquad r_k s^{q-k} - 1 = Q_k p, \qquad s^q - 1 = Np.$$

It now follows that p and s are relatively prime, p and r_k are relatively prime, N is the repetend of the repeating decimal for $1/p$, and that $p < s$. The first set of equations in (4) give the ordinary arithmetic process (P) for calculating the decimal $1/p$, and $q \geqq 2$ must be the smallest integer for the given p for which $r_q = 1$ in order to avoid the repetition of digit sequences in N. It now follows from the process (P) for developing $1/p$ that $r_k < p < s$ and we again find that no two of the multipliers, or residues, are equal. Thus (P) is also a process for finding q. We have also $3 \leqq p < s - 1$, and that p is not a divisor of $s - 1$.

In order to find all of the solutions for a given s we consider all the positive integers $p \geqq 3$, prime or composite, less than $s - 1$, prime to s and not divisors of $s - 1$. If no one of these values of p yields a solution by means of (P), then the problem has no solution. If (P) yields solutions we first obtain all such and then examine each as follows. If for a solution given by p, there exists a greatest positive integer $a > 1$ such that $ad_k < s$, $1 \leqq k \leqq q$, for this solution, then to this solution there correspond a solutions with the same q and the same set of multipliers r_k. If a' is an integer such that $2 \leqq a' \leqq a$ and a' is a divisor of p, $p = a'p'$, then this solution is given by (P) for p'. If a' is not a divisor of p, we have a solution which is not given by (P). When this is done for all of the (P) solutions we have all of the solutions of the problem.

The first value of s for which there is a solution is $s = 5$, and the following table gives the number n_s of solutions of both types for each s, $5 \leqq s \leqq 19$. The value $s = 19$ is the only one of the table which has a solution of the second type, a number of two digits 2(10) which is obtained from the solution of two digits 15 given by (P) for $p = 15$, with $r_1 = 4$, $r_2 = 1$.

s	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n_s	1	0	2	2	2	1	6	2	6	4	4	4	11	4	13

Necessary and sufficient conditions that there exists for a given s a solution of the second type are as follows:

There must be a p which gives a solution N of the first type with multipliers r_k, $1 \leqq k \leqq q$, $r_q = 1$, and there must be a positive integer r' less than p and prime to p which is not one of the q multipliers r_k for p. Finally we must have $r'r_k < p$, $1 \leqq k \leqq q$. Then $r'N$ is a solution of the second type with the same multipliers r_k as for N.

This theorem reduces the labor of constructing a table for n_s; its proof follows without difficulty from what is given above but seems long, and this is left to the reader. A large number of special cases of s having numbers of the second type may be obtained as follows: Let r' and a be two positive integers each greater than unity, and set $r_1 = r'a$, $r_k = r_1^k$, $1 \leqq k \leqq q-1$, $r_q = 1$, $q \geqq 2$, $p = r_1^q - 1$, $s = p + r_1$. Then an N of the first type is given by $d_k = r_1^{k-1}$, $1 \leqq k \leqq q-1$, $d_q = r_1^{q-1} + 1$, and $r'N$ is of the second type. If for example $q = 2, r' = 2, a = 4$, we get $p = 63 = 3 \cdot 21$, $s = 71$. This suggests trying $p = 21$, $s = 29$, $r' = 2$, $r_1 = 8$, and it turns out that this gives also a second type. By trial we find that no s, $19 < s < 29$, yields a second type.

The proposer remarked that the problem and solution result from theorems of his article in the *Gazeta Matematica*, Bucharest, 1935, pp. 385–392. A remark by Walter B. Carver states that the unique solution is the repetend of the decimal for 1/7. He stated also that, if the requirement $a_1 \neq 0$ is omitted, there are many solutions obtained similarly from primes p for which 10 is a primitive root, for example 1/17 gives a solution where $a_1 = 0$. He referred to the paper *On cyclic numbers*, by Solomon Guttman, in this MONTHLY [1934, 159]. The paper mentioned by Carver refers to an earlier one by Robert E. Moritz, in this MONTHLY [1927, 33], which begins with a remark that the properties of the number of the problem have been known for some years.

Squares within a Square

4014 [1941, 704]. *Proposed by P. Erdös, University of Pennsylvania*

Show that, if S_1 and S_2 are two squares contained in the unit square so that they have no point in common, the sum of their sides is less than unity.

It is very likely true that, if we have k^2+1 squares contained in the unit square so that no two of them have a point in common, the sum of their sides is less than k.

Solution by A. Seidenberg, Johns Hopkins University

If two squares S_1 and S_2 are contained in a unit square and have no point in common, then the sum of their sides is less than one.

Proof: Since the squares S_1 and S_2 have no point in common, a line l having no points in common with S_1 or S_2 can be drawn dividing the unit square into two parts U_1 and U_2, with U_1 containing S_1 and U_2 containing S_2. Moreover l may be taken so as to form with the two diagonals of the unit square a non-isosceles right triangle. Let P be the vertex of the acute angle of this triangle which is greater than 45 degrees. Then P must lie inside the unit square as other-

wise l would lie entirely outside the open unit square. Let P lie on the diagonal A_1A_2, A_1 in U_1, A_2 in U_2. Since the angle at P is greater than 45 degrees, the square with diagonal PA_1 lies inside U_1 and the square with diagonal PA_2 lies inside U_2. The theorem then follows from the inequalities:

$$\text{side of } S_i < \frac{\sqrt{2}}{2} PA_i, \qquad i = 1, 2.$$

These inequalities follow immediately from the following theorem.

THEOREM. *The square of maximum area contained in a right triangle has a vertex at the right angle of the triangle.*

Proof: Let the right triangle ABC have its vertices at $A=(a, 0)$, $B=(0, b)$, $C=(0, 0)$, $a>0$, $b>0$, in some rectangular cartesian coordinate system. Consider the set Σ of squares $RSTU$ such that R is on $y=0$, S is on $x=0$, and T is on $bx+ay-ab=0$, where RST is a right angle. Clearly, if there is a maximum square contained in ABC then it has a vertex on each side of the triangle and, for a proper lettering of the triangle, it may be supposed to be an element of Σ. By elementary considerations, one sees that S is equidistant from the line AB and the line m through R perpendicular to AB. Hence the equation of m if $S=(0, s)$ is $-ax+by-s(a+b)+ab=0$, so that R is the point $(r, 0)$ where

$$ar = -s(a+b)+ab \tag{1}$$

Let $R_1S_1T_1U_1$ be the element of Σ inside ABC for which $R=(0, 0)$. Let $R_2S_2T_2U_2$ be the element of Σ inside ABC for which T_2U_2 is on the line AB. These elements are unique because, for example, $S=(0, ab/(a+b))$ and $R=(ka, 0)$, $S=(0, kb)$ where $k(a^2+ab+b^2)=ab$.

When $s=ab/(a+b)$ then $r=0$; when $s=kb$, $r=ka$. Hence, because of the minus sign in (1),

$$r<0 \text{ if } s>ab/(a+b), \qquad r>ka \text{ if } s<kb.$$

In the first case, when $s>ab/(a+b)$, R is outside the triangle ABC because $r<0$, and in the second case, when $s<kb$, U is outside the triangle because the slope of TU is greater than that of AB. Thus $RSTU$ is inside ABC only if $kb \leqq s \leqq ab/(a+b)$.

By (1), $(RS)^2=r^2+s^2$ is quadratic in s, and since $(RS)^2 \to \infty$ as $s \to \infty$, $(RS)^2=d(s)$ attains a minimum (and not a maximum). Hence to obtain the maximum square $RSTU$ lying within ABC it remains only to compare $R_1S_1T_1U_1$ with $R_2S_2T_2U_2$. A simple inequality shows that $R_1S_1T_1U_1$ is the larger. This completes the proof.

Editorial Note. The auxiliary theorem may be proved in another way. If S is a square inside any given triangle ABC, a homothetic transformation with center A carries S into a square S_a with a vertex on BC which is the transform of the vertex of S nearest BC so that S_a lies inside ABC. Similarly using the cen-

ter B, the square S_a goes into S_b with a vertex on BC and another on CA; and finally, using the center C, S_b goes into S_c inside ABC with a vertex on each side, where $S \leqq S_a \leqq S_b \leqq S_c$.

Suppose now that C is a right angle and that the notation is such that $S_c = RSTU$ has the consecutive vertices R, S, T on CA, CB, AB, where angle $SRC = \theta$, $0 < \theta < \pi/2$, and a is the length of a side of S_c. Then the square of side a with the vertices U' on CA, R' at C, S' on BC has the fourth vertex T' actually inside ABC. For, with CA, CB as x, y axes the slopes of TT' and TU are respectively,

$$\frac{1 - \sin\theta - \cos\theta}{1 - \sin\theta}, \qquad -\frac{\sin\theta}{\cos\theta};$$

and the value of the left expression is less than that of the right since $\cos\theta(1-\sin\theta) < 1-\sin\theta$. Also T' is nearer CA than T, and hence T' lies actually inside ABC. Thus the maximum square inside ABC has one side along CA, another along CB, and the fourth vertex on AB.

A proof of the conjecture concerning k^2+1 squares, stated by the proposer in the second sentence of this problem 4014, will still be welcomed.

Tetrahedron with Variable Volume

4015 [1941, 704]. *Proposed by N. A. Court, University of Oklahoma*

If the base of a variable tetrahedron is fixed and the opposite vertex varies on a fixed sphere, the volume of the tetrahedron is numerically equal to the power of the variable vertex with respect to another fixed sphere.

Solution by C. E. Springer, University of Oklahoma

Let the vertices A_i of the tetrahedron have coordinates (x_i, y_i, z_i, t_i) $(i = 1, 2, 3, 4)$ with $t_i = 1$, and let A_1 be the variable vertex which lies on the fixed sphere $x^2+y^2+z^2 = r^2$, so that $x_1^2+y_1^2+z_1^2 = r^2$. If X_1, Y_1, Z_1, T_1, are the cofactors of x_1, y_1, z_1, t_1 in the determinant $|x_1y_2z_3t_4|$, the numerical value of the volume V is given by $6V = |x_1y_2z_3t_4| = x_1X_1+y_1Y_1+z_1Z_1+t_1T_1$, and the latter expression is six times the power of the point (x_1, y_1, z_1, t_1) with respect to the fixed sphere $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$, where

$$a = \mp X_1/12, \qquad b = \mp Y_1/12, \qquad c = \mp Z_1/12,$$
$$R^2 = a^2 + b^2 + c^2 + r^2 \mp T_1/6.$$

These two spheres may be both real, or one real and the other an imaginary or null sphere.

Solved also by the proposer.

The proposer's solution is based upon and follows easily from the theorem in his *Modern Pure Solid Geometry*, p. 184, art. 581; and he stated that the two desired spheres are both real if the plane of the base cuts the given sphere. He also stated that the analogous problem in the plane was considered in *Educational Times, Reprints*, vol. 48, 1893, p. 51, Q. 11294.

Squares with Distinct Digits

4020 [1942, 64]. *Proposed by V. Thébault, San Sebastian, Spain*

With three consecutive odd digits form a number of five digits whose square has ten distinct digits.

Solution by E. P. Starke, Rutgers University

Three digits of N are 1, 3, 5, or 3, 5, 7 or 5, 7, 9. N^2, being composed of the digits 0, 1, 2, $\cdots$, 9, is divisible by 9; whence N is a multiple of 3. Thus the digits of N must be 1, 3, 3, 3, 5 or 1, 1, 3, 5, 5, or 3, 3, 3, 5, 7 or 3, 5, 5, 7, 7 or 5, 7, 9, 9, 9 or 5, 5, 7, 7, 9. Permutations (150 in all) of these digits give those values of N formed from three consecutive odd digits whose squares consist of nine or ten digits and are divisible by 9. If, after discarding all of these which are less than 31622 ($<10^{9/2}$), one squares the others, the desired numbers are obtained by inspection. But since a square which presents duplication of digits in its first few and last few places may be eliminated, most of the labor of squaring may be avoided by comparison with a table of squares. Upon carrying out this program, one obtains:

$$35337^2 = 1248703569, \qquad 35757^2 = 1278563049, \qquad 75759^2 = 5739426081.$$

Editorial Note. The proposer gave the same results without indicating his method of derivation.

A Differential Operator

4021 [1942, 127]. *Proposed by Orrin Frink, Jr., Pennsylvania State College*

The differential operator D^2+1 may be factored in many ways; for example, it may be written $(D+\cot x)(D-\cot x)$, or $(\sec x \cdot D)(\cos x \cdot D+\sin x)$, or $(\sin x \cdot D+2\cos x)(D \csc x)$. Show that the most general method of expressing the differential operator $(D+a)^2+b^2$ as the product of two real first order differential operators is given by the formula

$$(D+a)^2+b^2 = r^{-1}[D+a-b\tan(bx+c)-r'/r]\cdot r[D+a+b\tan(bx+c)],$$

where a, b, and c are real numbers, and r is a differentiable function of x.

Solution by R. P. Agnew, Cornell University

The problem is that of characterizing functions p, q, r and s of x such that

$$[(D+a)^2+b^2]y = (pD+q)(rD+s)y \tag{1}$$

is, for each function $y(x)$ having two derivatives, an identity in x over some interval of values of x. Upon expanding the operators and equating powers of D, we obtain the relations

$$ps'+qs = a^2+b^2; \qquad pr'+ps+qr = 2a; \qquad pr = 1. \tag{2}$$

The existence of the derivatives r' and s' and the validity of the relations (2) may also be established by setting $y=1$, $y=x$, and $y=x^2$ successively in (1). From the third of equations (2), we see that $r\neq 0$ and $p=r^{-1}$. It turns out that

q and s can also be expressed in terms of r in the following way. Setting $p=r^{-1}$ in the first two of the equations (2) and eliminating q from the resulting equations gives

$$rs' - sr' = s^2 + 2ars + (a^2 + b^2)r^2. \tag{3}$$

Since $r\neq 0$, we can divide (3) by r^2 and set $u=s/r$ in the result to obtain $u'=u^2-2au+a^2+b^2$. In case $b\neq 0$, we obtain

$$\frac{d}{dx}\tan^{-1}\frac{u-a}{b} = \frac{bu'}{(u-a)^2+b^2} = b \tag{4}$$

and accordingly $u=a+b\tan(bx+c)$ where c is a constant. Thus

$$s = ru = r[a + b\tan(bx + c)]. \tag{5}$$

Putting the expressions for p and s in the second of equations (2) and solving for q gives

$$q = r^{-1}[a - b\tan(bx + c) - r'/r]. \tag{6}$$

Conversely, if (5) and (6) hold and $p=r^{-1}$, where r is a nonvanishing differentiable function and c is a constant, then the three equations (2) hold when x lies in an interval over which $\tan(bx+c)$ is defined. Thus, when $b\neq 0$, the result is as stated; an interval over which the factoring is valid is an interval over which $bx+c$ is not an odd multiple of $\pi/2$.

When $b=0$, the differential equation involving u becomes $u'=(u-a)^2$. The solution $u=a$ of this equation leads to the formula

$$(D + a)^2 = r^{-1}[D + a - r'/r]\cdot r[D + a]. \tag{7}$$

Over each interval in which u is differentiable and different from a, the equation $u'=(u-a)^2$ implies that

$$\frac{d}{dx}\frac{1}{u-a} = \frac{-1}{(u-a)^2}\frac{du}{dx} = -1 \tag{8}$$

and hence that $(u-a)^{-1}=c-x$ and $u=a+(c-x)^{-1}$ where c is a constant. On computing q and s for this case, we obtain the formula

$$(D + a)^2 = r^{-1}[D + a - (c - x)^{-1} - r'/r]\cdot r[D + a + (c - x)^{-1}] \tag{9}$$

where again r is a differentiable function and c is a constant. The factoring in (9) is valid over the infinite interval $x<c$ and over the infinite interval $x>c$. The factorings in (7) are obtained by setting $b=0$ in the factoring of the statement of the problem; those in (9) are not.

Solved also by E. A. Nordhaus, John Williamson, and the proposer. In these solutions the case of $b=0$ was not considered.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Applications for Benjamin Peirce Instructorships at Harvard University for the academic year 1943–1944 should be sent to the Chairman of the Department of Mathematics. Candidates should have received the Ph.D. degree or have had equivalent training.

Dr. G. A. Sarton of Harvard University delivered, on October 13, the Averill Lecture at Colby College.

Professor V. G. Grove of Michigan State College has resigned as chairman of the department of mathematics.

Professor E. R. Hedrick, retired vice-president of the University of California, and Professor Emeritus Virgil Snyder of Cornell University have been appointed visiting professors in mathematics at Brown University.

Assistant Professors M. R. Hestenes and W. T. Reid of the University of Chicago have been promoted to associate professorships.

Associate Professor C. G. Killen of Louisiana State Normal College has been promoted to a professorship.

Professor A. C. Lunn of the University of Chicago has retired after forty years of service.

Professor E. B. Mode has been made chairman of the department of mathematics at Boston University.

Professor E. E. Moots of Cornell College, Iowa, has been elected Associate Director of Research on Sound Control at Cruft Laboratory, Harvard University.

Dr. Haim Reingold is now in charge of the Mathematics Division of the Communication Engineering Courses of the Signal Corps at the Illinois Institute of Technology.

Assistant Professor M. F. Smiley of Lehigh University is now a Lieutenant (j.g.) in the U. S. Navy, and is teaching at the U. S. Naval Academy.

Dr. Alfred Tarski, formerly of the University of Warsaw, has joined the faculty of the University of California.

Dr. H. P. Thielman of the College of St. Thomas has been appointed an assistant professor at Iowa State College.

Professor A. D. Campbell, chairman of the department of mathematics of Syracuse University, was drowned in Green Lake, New York, on September 23, 1942. He was a charter member of the Mathematical Association.

Professor I. M. DeLong, since 1925 emeritus professor of mathematics at the University of Colorado, died September 2, 1942. He was a charter member of the Mathematical Association.

Dr. C. N. Haskins, Chandler Professor of Mathematics at Dartmouth College, died November 13, 1942, at the age of sixty-eight.

Professor J. F. Reilly of the State University of Iowa died August 18, 1942, at the age of sixty-six. He was a charter member of the Mathematical Association and served as secretary of the Iowa Section from 1921 to 1933.

FROM THE EDITOR

The Editor-in-Chief wishes to express his indebtedness to many persons during the first year of his editorship. At the beginning certain artistic and minor typographical improvements in the cover of the MONTHLY and in its internal arrangement were made. In this he was assisted by Miss Mary D. Alexander of the University of Chicago Press as well as by experts of the Banta Publishing Company.

For assistance in reading and appraising papers presented to the MONTHLY he wishes to express his appreciation of the work of the following:

A. A. Albert, R. C. Archibald, I. A. Barnett, L. M. Blumenthal, Louis Brand, H. E. Bray, H. E. Buchanan, Herbert Busemann, W. H. Bussey, W. B. Carver, R. V. Churchill, N. A. Court, A. B. Coble, H. S. M. Coxeter, C. J. Coe, L. W. Cohen, N. B. Conkwright, W. M. Davis, J. L. Doob, Otto Dunkel, Philip Franklin, Orrin Frink, Jr., M. R. Hestenes, R. V. Heath, Dunham Jackson, R. A. Johnson, L. C. Karpinski, A. J. Kempner, J. E. LaFon, R. E. Langer, N. H. McCoy, E. J. McShane, Karl Menger, W. E. Milne, J. R. Musselman, E. J. Moulton, Ivan Niven, Rufus Oldenburger, I. E. Perlin, H. B. Phillips, Tibor Radó, W. T. Reid, M. A. Sadowsky, Otto Schilling, W. S. Snyder, Norman Steenrod, M. H. Stone, R. J. Walker, H. S. Wall, J. H. Weaver, Marie J. Weiss, L. R. Wilcox and R. L. Wilder.

In an early issue of the MONTHLY a Department of War Information will be instituted. This will be under the editorship of Professor C. V. Newsom, University of New Mexico, Albuquerque, New Mexico, to whom all pertinent items should be sent.

SUGGESTIONS FROM THE OFFICE OF SCIENTIFIC PERSONNEL OF THE NATIONAL RESEARCH COUNCIL

As soon as the Army and Navy training programs are in full operation there will be an unprecedented demand for teachers of physics and of mathematics. The situation will be particularly critical in the field of physics where the teaching ranks of colleges and universities have already been seriously depleted.

It is the business of the Office of Scientific Personnel to assist in the placing of the scientific specialist where he can serve the war effort. Because the present supply of readily available physicists approximates zero and the supply of mathematicians is running low, perhaps this office can assist best by suggesting two sources close at hand to the institutions which will participate in the Army and Navy training programs.

The first source of supply is the nearby institutions which will not have Army or Navy training programs. Although these institutions should continue to teach physics to as many students as possible, some departments of physics will find it impossible to continue in operation and their staff members should be added to departments in need of their services.

The reason why all departments of physics which can possibly do so should continue in operation is that it is probable that only the needs of the Army and the Navy will be satisfied through the training programs. The further needs of war research, war industry, and teaching must be satisfied, for the most part, by women and others ineligible for military service. In this connection it should be mentioned that there seems to be no possibility of meeting the need for competent teachers of physics in the secondary schools.

Accordingly, it is important that colleges which are not fortunate enough to secure training contracts should continue to teach physics to even larger numbers than before. So great is the need for men and women trained in physics that every effort should be made to recruit into departments of physics all students with the necessary aptitude. There must be many students with aptitude for physics who do not normally find their way into this field.

These considerations make it clear that expanding departments should not take teachers from other institutions unless it is necessary for them to find employment elsewhere. The other source of supply, and the one that should be utilized wherever possible, is within the institution itself.

In any college or university there are teachers of other subjects, including such fields as botany, geology, physiology, psychology, and zoology (as well as the obvious fields of chemistry and engineering from which a few may be recruited), who have sufficient knowledge of physics so that, with a little brushing up and some observation of good physics teachers at work, they can become proficient teachers of beginning physics. These men and women should be found at once and encouraged to prepare themselves for the teaching service which they will almost certainly be called upon to perform either at their own institutions or elsewhere.

Still another consideration should be kept in mind by physics departments in making their adjustments to these new demands—all teaching and research in

physics must be directed to war ends. Advanced undergraduate and graduate work will soon make only light demands upon staff time except in the small number of institutions where the Army and Navy training programs will call for advanced work. Therefore, in many institutions there will be physicists who can serve the war effort better by going into war research than by teaching beginning physics. Institutions must release such men freely and replace them from the two sources already mentioned which promise to furnish an adequate supply. There seems to be no hope of being able to satisfy the present and future needs for war research and it is urged that all physicists who are as good research men as they are teachers of beginning physics should register with this office at once indicating when they will be available.

Everything that has been said with regard to physics applies, although to a lesser degree, in the field of mathematics.

It is the hope of this office that most of the required readjustments in the staffs of physics and mathematics departments may be worked out by drawing in new staff members from nearby institutions or by adjustments within the institutions themselves. This office will facilitate such adjustments in every way possible if they cannot be worked out locally and will be glad to receive information regarding institutional needs and available personnel. This will be done, however, only with the understanding that we will attempt to prevent bidding of institution against institution in mad competition for personnel. It is to the interest of all concerned that a sufficient supply of teachers of physics and mathematics should be developed and that the necessary readjustments be made easily and efficiently.

HOMER L. DODGE
Director, Office of Scientific Personnel
National Research Council

2101 Constitution Avenue
Washington, D. C.
January 11, 1943

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

FOR WAR COURSES IN MATHEMATICS

W. W. Hart and W. L. Hart's

Plane Trigonometry, Solid Geometry and Spherical Trigonometry

• ***The Plane Trigonometry*** **is an efficient, concise presentation of the fundamentals with a strong focus on numerical applications of importance in aeronautics and warfare •** ***The Solid Geometry*** **provides material suitable for a brief course or a review •** ***The Spherical Trigonometry*** **gives a rounded treatment of the usual theory together with extensive applications which supply an excellent background for the study of celestial navigation.** ***With Tables, 414 pages, $2.60; without Tables, 287 pages, $2.35.***

Plane Trigonometry with Applications

• **Consists of** ***Plane Trigonometry*** **from the preceding complete text with an added chapter on** ***plane and middle latitude sailing. With Tables, 304 pages, $2.00; without Tables, 177 pages, $1.75.***

Solid Geometry and Spherical Trigonometry

The solid geometry and spherical trigonometry sections from the complete text. ***With Tables, 250 pages, $2.15; without Tables, 123 pages, $1.90.***

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

MATHEMATICAL ASSOCIATION OF AMERICA

The office of the Secretary-Treasurer of the Mathematical Association, which has been at Oberlin, Ohio, since the organization in 1915, has been moved to Ithaca, New York, as of January 1, 1943, with the beginning of Professor Walter B. Carver's term as Secretary-Treasurer. All business correspondence, all subscriptions to the

AMERICAN MATHEMATICAL MONTHLY,

requests for change of address, and any applications for membership should therefore be sent to the following address:

MATHEMATICAL ASSOCIATION OF AMERICA
McGraw Hall, Cornell University
Ithaca, New York

THE CARUS MONOGRAPHS

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSORS DAVID EUGENE SMITH and JEKUTHIEL GINSBURG. (First Impression, 1934.)

No. 6. *Fourier Series and Orthogonal Polynomials,* by PROFESSOR DUNHAM JACKSON. (First Impression, 1941.)

No. 7. *Vectors and Matrices,* by PROFESSOR C. C. MACDUFFEE. (To appear in the spring of 1943.)

Price $1.25 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary, McGRAW HALL, Cornell University, ITHACA, N.Y.

Additional copies for members, and copies for non-members, may be purchased from the Open Court Publishing Co., La Salle, Illinois, at the regular price of $2.00 per copy.

Outline of the History of Mathematics

by RAYMOND CLARE ARCHIBALD

Fifth edition, June 1941, ii, 76 pages

THIS thoroughly revised and considerably enlarged edition is published by the Mathematical Association to meet a constant demand for up-to-date information on the History of Mathematics, not available in any other single English work. The syllabus, twenty pages of references to other material and sources, and an entirely new Index of Names, provide an excellent basis for a teacher to conduct a semester or year course in this field, or for a student wishing greatly to extend his knowledge. Earlier editions of the Outline have been reviewed very favorably throughout the world. Our expectation is that this fifth edition published within ten years, and more fundamentally revised and extended than any other edition, will continue to meet a need in this country and elsewhere.

Price 75 cents a copy, postpaid, remittance with order

No discount in price to anyone

MATHEMATICAL ASSOCIATION OF AMERICA

McGRAW HALL, CORNELL UNIVERSITY

ITHACA, NEW YORK

BRINK'S ALGEBRAS

By Raymond W. Brink, Ph.D., Professor of Mathematics, University of Minnesota

ALGEBRA: COLLEGE COURSE

Provides for a complete and rich course in college algebra. Cultivates logical habits of thought, develops a high degree of technical skill, and assures a mastery of principles. 8vo. 329 pp. $2.15.

COLLEGE ALGEBRA

Presents all the material in *Algebra: College Course* plus a systematic review of high-school higher algebra. A strong feature is its adaptability to courses of various lengths and purposes. 8vo. 445 pp. $2.40.

INTERMEDIATE ALGEBRA

Contains the review of high-school algebra included in *College Algebra* and supplies the needs of students with only one year of high-school algebra who wish to prepare for work in higher mathematics. 8vo. 268 pp. $1.50.

D. APPLETON-CENTURY COMPANY

35 W. 32nd St., New York — 2126 Prairie Ave., Chicago

Scripta Mathematica Publications

1. *Scripta Mathematica* is a quarterly journal devoted to the history and philosophy of mathematics. Subscription $3.00 per year.

2. *Scripta Mathematica Library.* Vol. I. *Poetry of Mathematics and other Essays,* by David Eugene Smith. Vol. II. *Mathematics and the Question of Cosmic Mind,* by Cassius Jackson Keyser. Vol. III. *Scripta Mathematica Forum Lectures.* Vol. IV. *Fabre and Mathematics and other Essays,* by Professor Lao G. Simons. Price of each volume in a beautiful silver-stamped cloth binding, $1.00. Vol. V. *Galois Lectures.* Price $1.25.

3. *Portraits of Eminent Mathematicians, Philosophers & Physicists with Their Biographies.* Portfolio I (13 folders), out of print. Portfolio II (14 folders), $3.75. Portfolio III (13 folders), $3.75. Portfolio IV (13 folders), $3.75.

4. *Visual Aids in the Teaching of Mathematics.* Single Portraits, mathematical themes in design, interesting curves and other pictorial items. Suitable for framing and for inclusion in students's notebook. List on request.

5. *George Peacock's Treatise on Algebra.* (Reprint) 2 Vols. $6.50.

SCRIPTA MATHEMATICA, Yeshiva College
Amsterdam Avenue and 186th Street, New York City

McGraw-Hill Books of Unusual Interest

VECTOR AND TENSOR ANALYSIS

By HOMER V. CRAIG, University of Texas. In press—read in January

The first part of this new book is concerned with the mathematical background; the second and third parts deal with vector and tensor analysis; the fourth part presents the applications (mechanics and relativity). Emphasis has been placed on the invariantive and transformation aspects of the subject.

DIFFERENTIAL EQUATIONS

By RALPH PALMER AGNEW, Cornell University. 341 pages, 6 x 9. $3.00

This book has been written for students with a reasonably good knowledge of elementary calculus, that they may master the techniques by which differential equations are obtained and solved and by which the solutions are applied. It has been the objective of the author to fill the gap between "problem books" and "theory books."

PRACTICAL MATHEMATICS FOR HOME STUDY. *New third edition*

By the late CLAUDE I. PALMER; and SAMUEL F. BIBB, Illinois Institute of Technology. 696 pages, pocket size. Textbook edition, $3.00

An ideal text for war training courses, this book has been brought completely up to date. As before, it gives a clear, simple, and usable treatment of the essentials of mathematics —arithmetic, geometry, algebra, and trigonometry. A feature of the book is the wealth of applications and problems covering hundreds of practical situations.

RADIO NAVIGATION FOR PILOTS

By COLIN H. MCINTOSH, American Airlines, Inc. 175 pages, 5½ x 8. $2.00

Presents radio navigation from an aircraft viewpoint, strictly limited to the practical treatment demanded by pilots. Among the topics dealt with are beam techniques, orientation methods, the instrument approach, the radio direction finding loop, loop navigation techniques, position fixes by D/F bearings, the running fix, and the radio compass.

PLANE AND SPHERICAL TRIGONOMETRY. *New second edition*

By LYMAN M. KELLS, WILLIS F. KERN and JAMES R. BLAND, U. S. Naval Academy. 401 pages, 6 x 9. $2.00. With Tables, $2.75

A simple, logically arranged text in which the topics are so spaced that each is mastered in its turn. Features of the book are the carefully graded exercises, the treatment of synthetic division, the chapter on the slide rule, the abridged methods of multiplication and division, and the proposed model form for logarithmic computation.

SPHERICAL TRIGONOMETRY WITH NAVAL AND MILITARY APPLICATIONS

By LYMAN M. KELLS, WILLIS F. KERN AND JAMES R. BLAND. 163 pages, 6 x 9. $1.50. With tables (including Haversines), $2.40

Represents a revision and expansion of the spherical trigonometry section of the authors' *Plane and Spherical Trigonometry*. The important applications of trigonometry to navigation and related topics are discussed and illustrated by numerous examples.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc,

330 West 42nd Street New York, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

2

MAR - 3 1943

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 50 NUMBER 2

CONTENTS

The Applications of Mathematics in Meteorology . . B. Haurwitz 77

Diagnostic Testing Program in Purdue University . . . . . M. W. Keller, D. R. Shreve and H. H. Remmers 85

On an Integral Transformation of General Circuit Theory. . . . . Bernhard Gross 90

Convergent Sequences of Probability Distributions. J. H. Curtiss 94

Boolean Algebra as an Introduction to Postulational Methods . . . . . E. R. Stabler 106

Section Meeting 110

Discussions and Notes . . . . . R. C. Yates, F. E. Hohn 112

Recent Publications 116

Problems and Solutions. 119

News and Notices 136

The War Policy Committee 138

Meetings of the Association and Its Sections 138

FEBRUARY 1943